

# $(q, t)$ -hook formula for Birds and Banners

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for VWP-series  ${}_{12}W_{11}$ .

## Abstract

We study Okada's conjecture on  $(q, t)$ -hook formula of general  $d$ -complete posets. Proctor classified  $d$ -complete posets into 15 irreducible ones. We try to give a case-by-case proof of Okada's  $(q, t)$ -hook formula conjecture using the symmetric functions. Here we give a proof of the conjecture for birds and banners in which we use Gasper's identity for VWP-series  ${}_{12}W_{11}$ .

## 1 Introduction and the main results

The aim of this paper is to prove Okada's multivariate hook formula conjecture for birds and banners, i.e., Theorem 1.9. His conjecture is for general  $d$ -complete posets, and here we give a partial proof for birds and banners only. Proctor [9] has classified  $d$ -complete posets into 15 irreducible classes. Okada [8] has made his conjecture for general  $d$ -complete posets. he has proven two cases in his paper and we settle two cases in this paper so that the rest 11 classes are still left. Even though we do a case-by-case proof, we need the Macdonald polynomials and Gasper's identity for very well-poised series  ${}_{12}W_{11}$  in our proof. This paper is composed as follows. In this section we recall the fundamental concepts on  $d$ -complete partitions, and then state our main result, i.e., Theorem 1.9. To state Okada's conjecture we need the terminologies on  $d$ -complete posets. In Section 2 we recall the Macdonald polynomials. In Section 3 we rewrite Okada's conjecture by the Macdonald polynomials and use the fact that the Macdonald polynomials are basis for the ring of the symmetric functions. In Section 4 we prove the Macdonald polynomial identities obtained in Section 3 using Gasper's identity.

Let  $\mathbb{N}$  (resp.  $\mathbb{Z}$ ) be the set of nonnegative integers (resp. integers). Throughout this paper we use the standard notation for  $q$ -series (see [1, 3, 4, 5]):

$$(a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k), \quad (a; q)_n = \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}}$$

for any integer  $n$ . Usually  $(a; q)_n$  is called the  $q$ -shifted factorial, and we frequently use the compact notation:

$$(a_1, a_2, \dots, a_r; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n.$$

The  ${}_{r+1}\phi_r$  basic hypergeometric series is defined by

$${}_{r+1}\phi_r \left( \begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_n}{(q, b_1, \dots, b_r; q)_n} z^n. \quad (1.1)$$

A basic hypergeometric series  ${}_{r+1}\phi_r$  is said to be *balanced* if it satisfies  $qa_1 \cdots a_{r+1} = b_1 \cdots b_r$  and  $z = q$ , *well-poised* if it satisfies  $qa_1 = a_2b_1 = \cdots = a_{r+1}b_r$ , *very well-poised* if it is well-poised and satisfies  $b_1 = a_1^{\frac{1}{2}}$  and  $b_2 = -a_1^{\frac{1}{2}}$  (see [3, §2.1]). If  ${}_{r+1}\phi_r$  is very well-poised series, we use the notation

$${}_{r+1}W_r(a_1; a_4, \dots, a_{r+1}; q, z) = {}_{r+1}\phi_r \left[ \begin{matrix} a_1, qa_1^{\frac{1}{2}}, -qa_1^{\frac{1}{2}}, a_4, \dots, a_{r+1} \\ a_1^{\frac{1}{2}}, -a_1^{\frac{1}{2}}, qa_1/a_4, \dots, qa_1/a_{r+1} \end{matrix} ; q, z \right].$$

**Proposition 1.1.** Gasper's identity ([2, p.1065, (3.2)], [3, pp.250, Ex.8.15]) reads as follows:

$$\begin{aligned} {}_4\phi_3 \left[ \begin{matrix} a, b, c, d \\ bq/a, cq/a, dq/a \end{matrix} ; q, \frac{q^2}{a^2} \right] &= \frac{(a/d, bq/d, cq/d, abc/d; q)_\infty}{(q/d, ab/d, ac/d, bcq/d; q)_\infty} \\ &\times {}_{12}W_{11} \left( \frac{bc}{d}; \left( \frac{bcq}{ad} \right)^{\frac{1}{2}}, -\left( \frac{bcq}{ad} \right)^{\frac{1}{2}}, q \left( \frac{bc}{d} \right)^{\frac{1}{2}}, -q \left( \frac{bc}{d} \right)^{\frac{1}{2}}, \frac{ab}{d}, \frac{ac}{d}, a, b, c; q, \frac{q}{a} \right), \end{aligned} \quad (1.2)$$

where at least one of  $a, b, c$  is of the form  $q^{-n}$  ( $n = 0, 1, \dots$ ).

We use the notation in [8]. For nonnegative integers  $n$  and  $m$  we write

$$f(n; m) = f_{q,t}(n; m) = \frac{(t^{m+1}; q)_n}{(t^m; q)_n},$$

and

$$F(x) = F(x; q, t) = \frac{(tx; q)_\infty}{(x; q)_\infty},$$

where  $q$  and  $t$  are parameters and  $x$  is a variable (see [8, (5)(6)]). Hereafter we use the convention that  $f_{q,t}(n; m) = 0$  for a negative integer  $n < 0$ .

We use the notation in [7, 12] for partitions. Let  $\lambda = (\lambda_1, \lambda_2, \dots)$  be a partition, i.e.,  $\lambda_1 \geq \lambda_2 \geq \dots$  with finitely many  $\lambda_i$  unequal to zero. The length and weight of  $\lambda$ , denoted by  $\ell(\lambda)$  and  $|\lambda|$ , are the number and sum of the non-zero  $\lambda_i$  respectively. When  $|\lambda| = N$  we say that  $\lambda$  is a partition of  $N$ , and the unique partition of zero is denoted by  $\emptyset$ . The multiplicity of the part  $i$  in the partition  $\lambda$  is denoted by  $m_i(\lambda)$ . We identify a partition with its diagram (Ferrers graph)

$$D(\lambda) = \{ (i, j) \in \mathbb{Z}^2 : 1 \leq j \leq \lambda_i \}. \quad (1.3)$$

The conjugate  $\lambda'$  of  $\lambda$  is the partition obtained by reflecting the diagram of  $\lambda$  in the main diagonal. A partition is said to be *strict* if we have strict inequalities  $\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0$  with  $r = \ell(\lambda)$ . If  $\lambda$  is a strict partition, then its shifted diagram is defined by

$$S(\lambda) = \{ (i, j) \in \mathbb{Z}^2 : i \leq j \leq \lambda_i + i - 1 \}. \quad (1.4)$$

Hereafter we may use the same symbol  $\lambda$  to represent its diagram (or shifted diagram).

We use standard notation and terminology of [12, Chapter 3] related to posets. We write  $x < y$  if  $x$  is covered by  $y$ , i.e.,  $x < y$  and there is no  $z \in P$  such that  $x < z < y$ . A Hasse diagram is a diagram in which one represents each element of  $P$  as a vertex in the plane and draws an edge that goes upward from  $x$  to  $y$  whenever  $y$  covers  $x$ .

**Definition 1.2.** ([11], [12, §3.15]) Let  $P$  be a poset. A  $P$ -partition is a map  $\pi : P \rightarrow \mathbb{N}$  satisfying

$$x \leq y \text{ in } P \implies \pi(x) \geq \pi(y) \text{ in } \mathbb{N}. \quad (1.5)$$

Let  $\mathcal{A}(P)$  denote the set of  $P$ -partitions.

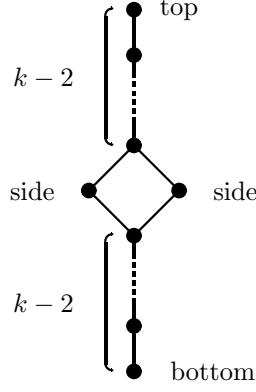


Figure 1: A double-tailed diamond poset  $d_k(1)$

First, we review the definition and some properties of  $d$ -complete posets. (See [9, 10].) For  $k \geq 3$ , we denote by  $d_k(1)$  the poset consisting of  $2k - 2$  elements, called *double-tailed diamond poset*, with the Hasse diagram depicted in Figure 1. The two incomparable elements are called the *sides*, the  $k - 2$  elements above them are called *neck* elements, and the maximum and minimum elements are called *top* and *bottom* respectively. If  $k = 3$  then we call  $d_3(1)$  a *diamond*. Let  $P$  be a poset. An interval  $[w, v] = \{x \in P : w \leq x \leq v\}$  is called a  $d_k$ -interval if it is isomorphic to  $d_k(1)$ . A  $d_k^-$ -interval ( $k \geq 4$ ) is an interval isomorphic to  $d_k(1) - \{\text{top}\}$ . A  $d_3^-$ -interval consists of three elements  $x, y$  and  $w$  such that  $w$  is covered by both  $x$  and  $y$ . A poset  $P$  is *d-complete* if it satisfies the following three conditions for every  $k \geq 3$ :

- (D1) If  $I$  is a  $d_k^-$ -interval, then there exists an element  $v$  such that  $v$  covers the maximal elements of  $I$  and  $I \cup \{v\}$  is a  $d_k$ -interval.
- (D2) If  $I = [w, v]$  is a  $d_k$ -interval and the top  $v$  covers  $u$  in  $P$ , then  $u \in I$ .
- (D3) There are no  $d_k^-$ -intervals which differ only in the minimal elements.

We quote a proposition due to Proctor [9, Proposition in §3] (also see [8, Proposition 4.1]):

**Proposition 1.3.** ([9, Proposition in §3]) Let  $P$  be a  $d$ -complete poset. Suppose that  $P$  is connected, i.e., the Hasse diagram of  $P$  is connected. Then we have

- (a)  $P$  has a unique maximal element  $v_0$ .
- (b) For each  $v \in P$ , every saturated chain from  $v$  to the maximum element  $v_0$  has the same length.

Hence  $P$  admits a rank function  $r : P \rightarrow \mathbb{N}$  such that  $r(x) = r(y) + 1$  if  $x$  covers  $y$ .

A *rooted tree* is a poset which has a unique maximal element, and is such that each non-maximal element is covered by exactly one other element. Let  $P$  be a poset with a unique maximal element. The top tree  $T$  of  $P$  is the filter (i.e.,  $x \in T$  and  $y \geq x$  implies  $y \in T$ ) of  $P$ , whose vertex set consists of all elements  $x \in P$  such that every  $y \geq x$  is covered by at most one other element of  $P$ .  $T$  is clearly a rooted tree and an element of  $T$  is called *top tree element*. Afterwards we use a particular kind of rooted tree. Let  $f \geq 0$  and  $h \geq g \geq 0$  be integers. The rooted tree  $Y(f; g, h)$  consists of one branch element above which a chain of  $f$  elements has been adjoined and below which two non-adjacent chains with  $g$  and  $h$  elements, respectively.

Let  $P$  be a connected  $d$ -complete poset with top tree  $T$ . An element  $x \in P$  is said to be *acyclic* if  $x \in T$  and it is not in the neck of any  $d_k$ -interval for any  $k \geq 3$ . An element of  $P$  is said to be *cyclic* if it is not acyclic. Let  $Q$  be a  $d$ -complete poset containing an acyclic element  $y$ . Let

$P$  be a connected  $d$ -complete poset. By Proposition 1.3 (a), let  $x$  denote the unique maximal element of  $P$ . Then the *slant sum* of  $Q$  with  $P$  at  $y$ , denoted  $Q^y \setminus_x P$ , is the poset formed by creating a covering relation  $x < y$ . A  $d$ -complete poset  $P$  is *slant irreducible* if it is connected and it cannot be expressed as a slant sum of two non-empty  $d$ -complete posets. Suppose that  $P$  is a connected  $d$ -complete poset with top tree  $T$ . An edge  $x < y$  of  $P$  is a *slant edge* if  $x, y \in T$  and  $y$  is acyclic. In [9] Proctor proves  $P$  is slant irreducible if and only if it contains no slant edges. Also,  $P$  is slant irreducible if and only if every acyclic element is a minimal element of its top tree. ([9, Proposition C of §4]) Given any connected  $d$ -complete poset  $P$ , first locate all of its slant edges. These may be erased in any order to produce a collection  $P_1, P_2, \dots$  of uniquely determined smaller non-adjacent connected  $d$ -complete posets. No new slant edges are created, and so each of  $P_1, P_2, \dots$  are slant irreducible. We say that  $P_1, P_2, \dots$  are the *slant irreducible components* of  $P$ . If  $P$  is an irreducible component, then its top tree  $T$  is of the form  $Y(f; g, h)$  for some  $f \geq 0$  and  $h \geq g \geq 1$  ([9, Theorem of §5]). In the paper he establish the following theorem, which describe the structure of any connected  $d$ -complete poset.

**Proposition 1.4.** (Proctor [9, Theorem in §4]) Let  $P$  be a connected  $d$ -complete poset. It may be uniquely decomposed into a slant sum of one element posets and irreducible components. The top tree of  $P$  is an analogous slant sum of the top trees of the irreducible components.

In §7 of [9] Proctor defines 15 disjoint classes of irreducible components  $\mathcal{C}_1, \dots, \mathcal{C}_{15}$  and have shown that these 15 disjoint classes exhaust the set of all irreducible components. For the list of 15 classes of irreducible  $d$ -complete posets see [9, Table 1]. The diagram (1.3) of an ordinary partition  $\lambda$  or the shifted diagram (1.4) of a shifted partition  $\lambda$  is regarded as a poset by defining its order structure as

$$(i_1, j_1) \geq (i_2, j_2) \iff i_1 \leq i_2 \text{ and } j_1 \leq j_2. \quad (1.6)$$

By this order the poset represented by a diagram  $P = D(\lambda)$  is called a *shape* with its top tree  $T = Y(f; g, h)$  where  $f = 0$ ,  $g = \ell(\lambda)$  and  $h = \ell(\lambda')$ . We use  $\mathcal{C}_1$  to express the class of shapes which is a class of irreducible  $d$ -complete posets defined in [9].

Another important class  $\mathcal{C}_2$  is the set of posets  $P = S(\alpha)$  of shifted diagrams for strict partitions  $\alpha$ , which is called *shifted shapes* with its top tree  $T = Y(f, g, h)$  where  $f = g = 1$  and  $h = \ell(\alpha)$ . Its Hasse diagram is designated by Figure 2 in which the first row has  $\alpha_1$  vertices, the second row  $\alpha_2$  vertices and so on. When depicting these posets as a Hasse diagram, we use the convention that a northwest vertex is larger than another in southeast. Here the larger dots and the heavier edges indicate the top tree. For later use we denote by  $P = P_2(\alpha)$  the Shifted shape associated with a strict partition  $\alpha$ . If  $P = P_2(\alpha)$  is the shifted shape associated with a strict partition  $\alpha$ ,

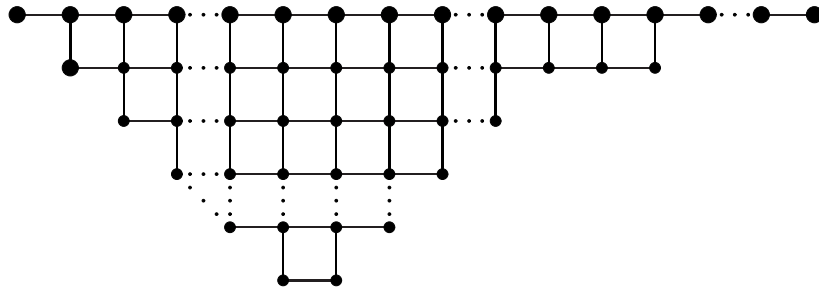


Figure 2: Shifted shapes  $\mathcal{C}_2$

then  $P$ -partition

$$\pi = (\pi_{ij})_{(i,j) \in S(\alpha)} \quad (1.7)$$

satisfies

$$\pi_{ij} \leq \pi_{i+1,j}, \quad \pi_{ij} \leq \pi_{i,j+1}, \quad (1.8)$$

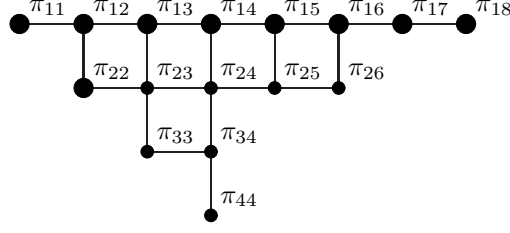


Figure 3:  $P$ -partition for shifted shape  $(8, 5, 2, 1)$

whenever the both sides defined. For example, Figure 3 is a  $P$ -partition for shifted shape  $(8, 5, 2, 1)$ .

In this paper we mainly consider only two classes, i.e., birds  $\mathcal{C}_3$  (Figure 4) and banners  $\mathcal{C}_6$  (Figure 6). Let  $\alpha = (\alpha_1, \alpha_2)$  and  $\beta = (\beta_1, \beta_2)$  be strict partitions such that  $\alpha_1 > \alpha_2 > 0$  and

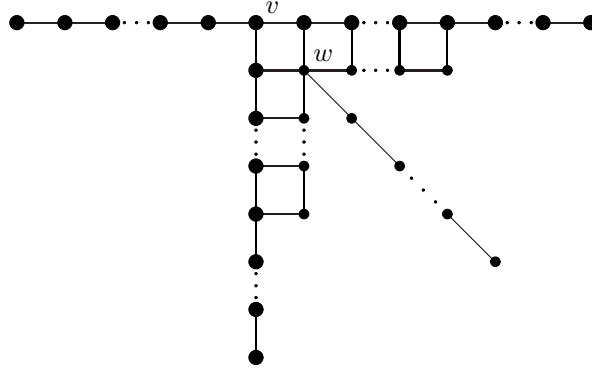


Figure 4: Birds  $C_3$

$\beta_1 > \beta_2 > 0$ . Define the bird  $P = P_3(\alpha, \beta; f)$  by

$$P = P_H \cup P_R \cup P_L \cup P_T$$

where

$$\begin{aligned} P_H &= \{ (1, j) : -f + 1 \leq j \leq 1 \}, \\ P_R &= \{ (i, j) : i \leq j \leq \alpha_i + i - 1 \ (i = 1, 2) \}, \\ P_L &= \{ (i, j) : j \leq i \leq \beta_j + j - 1 \ (j = 1, 2) \}, \\ P_T &= \{ (i, i) : 2 \leq i \leq f + 2 \} \end{aligned}$$

as a set and we regard it as a poset by defining its order structure (1.6) if and only if the both of  $(i_1, j_1)$  and  $(i_2, j_2)$  are in  $P_H \cup P_R \cup P_L$  or in  $P_T$  (see [9, Table 1 and Figure 5.3]). We call  $P_H$  the *head*,  $P_T$  the *tail*,  $P_R$  (resp.  $P_L$ ) the *right* (resp. *left*) *wing* of  $P$ . The Hasse diagram of a bird is as in Figure 4. Strictly speaking, we have to impose the condition  $\alpha_1 = \alpha_2 + 1$  and  $\beta_1 = \beta_2 + 1$  to let  $P$  be slant irreducible, but here we don't need this condition. For example, the left-picture in Figure 5 stands for  $P = P_3((4, 3), (4, 2); 2)$ . We have the chain  $[v, v_2]$  (resp.  $[w_2, w]$ ), which is the head (resp. tail) of  $P$ . Recall that a  $P$ -partition  $\pi$  satisfies the condition (1.5). When  $P = P_3(\alpha, \beta; f)$ , we associate the quadruple  $(\sigma, \tau; \rho, \theta)$  with  $\pi$ , where

$$\sigma = (\sigma_{i,j})_{(i,j) \in P_R}, \quad \tau = (\tau_{i,j})_{(j,i) \in P_L}, \quad \rho = (\rho_i)_{i=0, \dots, f}, \quad \theta = (\theta_i)_{i=0, \dots, f}$$

with

$$\begin{aligned} \sigma_{i,j} &= \pi(i, j) & \text{for } (i, j) \in P_R, & & \tau_{i,j} &= \pi(j, i) & \text{for } (i, j) \in P_L, \\ \rho_{-i+1} &= \pi(1, i) & \text{for } (1, i) \in P_H, & & \theta_{i-2} &= \pi(i, i) & \text{for } (i, i) \in P_T. \end{aligned} \quad (1.9)$$

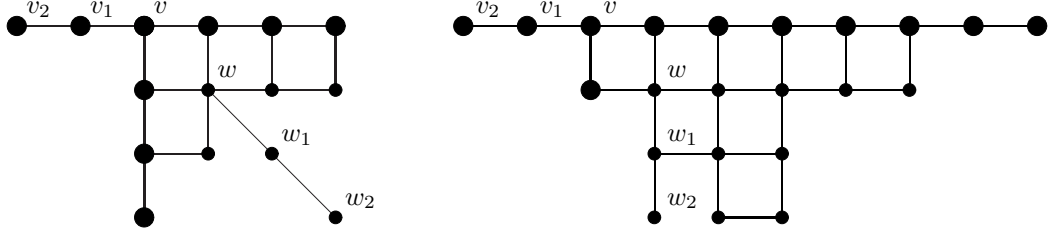


Figure 5: Bird  $P = P_3((4, 3), (3, 2); 2)$  and banner  $P = P_6((9, 6, 3, 2); 2)$

Hence we use the convention that  $\rho_0 = \sigma_{11} = \tau_{11}$  and  $\theta_0 = \sigma_{22} = \tau_{22}$ . We write  $\pi = (\sigma, \tau; \rho, \theta)$  hereafter. If  $P = P_3((4, 3), (4, 2); 2)$  then  $\pi$  is as the left picture of Figure 7.

Let  $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$  be a strict partition such that  $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4 > 0$ , and let  $f \geq 2$  be a positive integer. Let  $P$  be the set  $P = P_H \cup P_W \cup P_T$  of lattice points in  $\mathbb{Z}^2$ , where

$$\begin{aligned} P_H &= \{ (1, j) : -f + 2 \leq j \leq 1 \}, \\ P_W &= \{ (i, j) : i \leq j \leq \alpha_i + i - 1 \ (i = 1, 2, 3, 4) \}, \\ P_T &= \{ (i, 3) : 3 \leq i \leq f + 2 \}. \end{aligned}$$

We regard  $P$  as a poset by defining the order relation (1.6) if both of  $(i_1, j_1)$  and  $(i_2, j_2)$  are in  $P_H \cup P_W$  or in  $P_T$ , and call it a *banner* (see [9, Table 1 and Figure 5.6]). The Hasse diagram of a bird is as in Figure 4 in general. Strictly speaking again, we have to impose the condition  $\alpha_1 = \alpha_2 + 1$  to let  $P$  be slant irreducible, but we don't need this condition here. We call  $P_H$  the *head*,  $P_T$  the *tail*, and  $P_W$  the *wing* of  $P$ . We use the symbol  $P = P_6(\alpha; f)$  to mean the banner associated with a strict partition  $\alpha$  and a positive integer  $f$ . The Hasse diagram of a banner is given in Figure 6. For example, the right picture in Figure 5 stands for  $P = P_6((9, 6, 3, 2); 2)$ . If

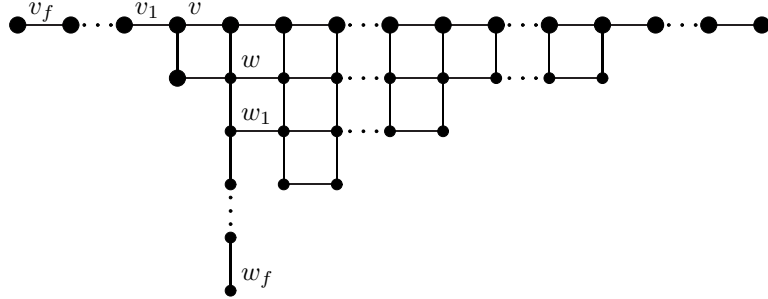


Figure 6: Banners  $C_6$

$P = P_6(\alpha; f)$  is the banner, we associate a triplet  $(\sigma; \rho, \theta)$  with a  $P$ -partition  $\pi$ , in which each component  $\sigma = (\sigma_{i,j})_{(i,j) \in S(\alpha)}$ ,  $\rho = (\rho_i)_{i=1, \dots, f}$ ,  $\theta = (\theta_i)_{i=1, \dots, f}$  are defined by

$$\begin{aligned} \sigma_{i,j} &= \pi(i, j) & \text{for } (i, j) \in P_W, \\ \rho_i &= \pi(1, -i + 2) & \text{for } i = 1, \dots, f, \quad \theta_i = \pi(i + 2, 3) \quad \text{for } i = 1, \dots, f. \end{aligned} \quad (1.10)$$

Hence we have  $\rho_1 = \sigma_{11}$  and  $\theta_1 = \sigma_{33}$ . Hereafter we write  $\pi = (\sigma; \rho, \theta)$ . For example, the right picture of Figure 7 is a  $P_6((9, 6, 3, 2); 2)$ -partition.

Let  $P$  be a connected  $d$ -complete poset and  $T$  its top tree. Let  $C$  be a set, called a *set of colors*, whose cardinality is the same as  $T$ . A *coloring* of  $P$  a coloring map  $c$  of  $P$  to the set of colors  $C$ .  $P$  is said to be *properly colored* if the coloring map  $c$  satisfies

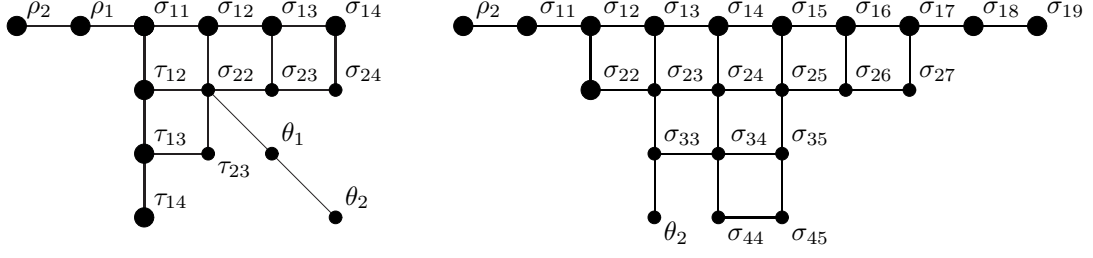


Figure 7:  $P$ -partitions

(C1)  $c(x) \neq c(y)$  if  $x$  and  $y$  are incomparable,

(C2)  $c(x) \neq c(y)$  if  $x$  covers  $y$ .

It is *simply colored* if, in addition:

(C3) whenever an interval  $[w, v]$  is a chain, the colors of the elements  $c(x)$  in the interval  $[w, v]$  are distinct.

If  $P$  is a rooted tree, then it is simply colored by the identity map  $P \rightarrow P$ , i.e. we assign a distinct color to each vertex of  $P$ .

**Proposition 1.5.** ([10, Proposition 8.6]) Let  $P$  be a connected  $d$ -complete poset and  $T$  its top tree. Let  $C$  be a set whose cardinality is the same as  $T$ . Then a bijection  $c : T \rightarrow C$  can be uniquely extended to a proper coloring  $c : P \rightarrow C$  satisfying the following condition:

(C4) If  $[w, v]$  is a  $d_k$ -interval then  $c(w) = c(v)$ .

Such a map  $c : P \rightarrow I$  is called a  $d$ -complete coloring.

For example, in the both picture of Figure 5 because  $[w_2, v_2]$  (resp.  $[w_1, v_1]$ ,  $[w, v]$ ) is a  $d_5$ -interval (resp.  $d_4$ -interval,  $d_3$ -interval),  $w_2$  (resp.  $w_1$ ,  $w$ ) and  $v_2$  (resp.  $v_1$ ,  $v$ ) have the same color. In Figure 6  $v_1$  (resp.  $v_2$ ) and  $v_3$  (resp.  $v_4$ ) have the same color since  $[v_3, v_1]$  (resp.  $[v_4, v_2]$ ) is a  $d_4$ -interval, however, the  $v_1$  and  $v_2$  have distinct colors since the both are in the top tree.

**Proposition 1.6.** (1) If  $\alpha$  is a strict partition with  $\text{length} \geq 2$ , then the top tree of the shifted shape  $P = P_2(\alpha)$  is given by

$$T = \{(1, j) : 1 \leq j \leq \alpha_1\} \cup \{(2, 2)\}, \quad (1.11)$$

and a  $d$ -complete coloring  $c : P \rightarrow \{0, 0', 1, 2, \dots, \alpha_1 - 1\}$  is given by

$$c(i, j) = \begin{cases} j - i & \text{if } i < j, \\ 0 & \text{if } i = j \text{ and } i \text{ is odd,} \\ 0' & \text{if } i = j \text{ and } i \text{ is even.} \end{cases} \quad (1.12)$$

Hence we see that  $P$  has the top tree  $Y(1; 1, \alpha_1 - 1)$ .

(2) If  $\alpha$  and  $\beta$  are strict partitions with  $\text{length} = 2$  and  $f \geq 1$  then the top tree of the bird  $P = P_3(\alpha, \beta; f)$  is given by

$$T = \{(1, j) : -f + 1 \leq j \leq \alpha_1\} \cup \{(i, 1) : 1 \leq i \leq \beta_1\}, \quad (1.13)$$

and a  $d$ -complete coloring  $c : P \rightarrow \{-f, \dots, -1, 0, 1, 2, \dots, \alpha_1 - 1\} \cup \{1', 2', \dots, (\beta_1 - 1)'\}$  is given by

$$c(i, j) = \begin{cases} j - i & \text{if } i < j, \text{ i.e., } (i, j) \in P_R, \\ (i - j)' & \text{if } 1 \leq j < i, \text{ i.e., } (i, j) \in P_L, \\ j - 1 & \text{if } i = 1 \text{ and } j \leq 1, \text{ i.e., } (i, j) \in P_H, \\ -i + 2 & \text{if } i = j \geq 2, \text{ i.e., } (i, j) \in P_T. \end{cases} \quad (1.14)$$

Hence we see that  $P$  has the top tree  $Y(f; \alpha_1 - 1, \beta_1 - 1)$ .

- (3) If  $\alpha$  is a strict partitions with length= 4 and  $f \geq 2$  then the top tree of the banner  $P = P_3(\alpha; f)$  is given by

$$T = \{(1, j) : -f + 2 \leq j \leq \alpha_1\} \cup \{(2, 2)\}, \quad (1.15)$$

and a  $d$ -complete coloring  $c : P \rightarrow \{-f + 1, \dots, -1, 0, 1, 2, \dots, \alpha_1 - 1\} \cup \{0'\}$  is given by

$$c(i, j) = \begin{cases} j - i & \text{if } i \neq j, \\ 0 & \text{if } i = j \text{ and } i \text{ is odd,} \\ 0' & \text{if } i = j \text{ and } i \text{ is even.} \end{cases} \quad (1.16)$$

Hence we see that  $P$  has the top tree  $Y(f; \alpha_1 - 1, 1)$ .

**Proof.** (1) is obtained in [8, Example 4.3(b)]. (2) and (3) are also obtained from (C1)–(C4).  $\square$

Let  $P$  be a connected  $d$ -complete poset and  $c : P \rightarrow C$  a  $d$ -complete coloring. Let  $z_i$  ( $i \in C$ ) be indeterminates. For a  $P$ -partition  $\pi \in \mathcal{A}(P)$ , we write

$$z^\pi = \prod_{v \in P} z_{c(v)}^{\pi(v)}.$$

As in [8, p.412] we associate a monomial  $z[H_P(v)]$  to each  $v \in P$ , called the *hook monomial*, which is uniquely determined by induction as follows:

- (a) If  $v$  is not the top of any  $d_k$ -interval, then we define

$$z[H_P(v)] = \prod_{w \leq v} z_{c(w)}.$$

- (b) If  $v$  is the top of a  $d_k$ -interval  $[w, v]$ , then we define

$$z[H_P(v)] = \frac{z[H_P(x)] \cdot z[H_P(y)]}{z[H_P(w)]},$$

where  $x$  and  $y$  are the sides of  $[w, v]$ .

Further we denote  $z[H_p] = \{z[H_P(v)] : v \in P\}$  the set of the hook monomials, and let  $F(z[H_p]; q, t)$  denote the product of  $F(z[H_P(v)]; q, t)$  over  $v \in P$ , i.e.,

$$F(z[H_p]; q, t) = \prod_{v \in P} F(z[H_P(v)]; q, t).$$

Let  $P$  be a connected  $d$ -complete poset with the maximum element  $v_0$ , and the rank function  $r : P \rightarrow \mathbb{N}$ . Let  $T$  be the top tree of  $P$ . Take  $T$  as a set of colors and let  $c : P \rightarrow T$  be the  $d$ -complete coloring such that  $c(v) = v$  for all  $v \in T$ . Let  $\hat{P} = P \sqcup \{\hat{1}\}$  be the extended poset, where  $\hat{1}$  is the new maximum element of  $\hat{P}$  which covers  $v_0$ . Then  $\hat{P}$  has its top tree  $\hat{T} = T \sqcup \{\hat{1}\}$ , where  $\hat{c} : \hat{P} \rightarrow \hat{T}$  with  $\hat{c}(\hat{1}) = \hat{1}$ .

**Definition 1.7.** Given a  $P$ -partition  $\pi \in \mathcal{A}(P)$ , let  $\hat{\pi} : \hat{P} \rightarrow \mathbb{N}$  be the extensions of  $\pi$  defined by  $\hat{\pi}(\hat{1}) = 0$ . Define a weight  $W_P(\sigma; q, t)$  by putting

$$W_P(\pi; q, t) = \frac{\prod_{\substack{x, y \in \hat{P} \\ x < y, \hat{c}(x) \sim \hat{c}(y)}} f(\pi(x) - \pi(y); d(x, y))}{\prod_{\substack{x, y \in P \\ x < y, c(x) = c(y)}} f(\sigma(x) - \sigma(y); e(x, y)) f(\sigma(x) - \sigma(y); e(x, y) - 1)}, \quad (1.17)$$

where  $\hat{c}(x) \sim \hat{c}(y)$  means that  $\hat{c}(x)$  and  $\hat{c}(y)$  are adjacent to each other in  $\hat{T}$ , and

$$d(x, y) = \frac{r(y) - r(x) - 1}{2}, \quad e(x, y) = \frac{r(y) - r(x)}{2}.$$

Note that if  $c(x) \sim c(y)$  then  $r(y) - r(x)$  is odd, and if  $c(x) = c(y)$  then  $r(y) - r(x)$  is even, hence  $d(x, y)$  and  $e(x, y)$  are nonnegative integers.



Now we quote Okada's  $(q, t)$ -hook formula conjecture.

**Conjecture 1.8.** (Okada [8]) Let  $P$  be a connected  $d$ -complete poset. Using the notations defined above, we have

$$\sum_{\pi \in \mathcal{A}(P)} W_P(\pi; q, t) z^\pi = F(z[H_P]; q, t). \quad (1.18)$$

Okada has proven this conjecture for shapes and shifted shapes. The purpose of this paper is to prove his conjecture for birds and banners.

**Theorem 1.9.** Okada's  $(q, t)$ -hook formula conjecture is true for birds and banners.

Given a  $P$ -partition  $\pi \in \mathcal{A}(P)$  for the shifted shape  $P = P_2(\alpha)$  for a strict partition  $\alpha$ , we write

$$f_\alpha^{\text{ND}}(\pi; q, t) = \prod_{\substack{(i,j) \in \alpha \\ i < j}} \prod_{m \geq 0} \frac{f(\pi_{i,j} - \pi_{i-m,j-m-1}; m) f(\pi_{i,j} - \pi_{i-m-1,j-m}; m)}{f(\pi_{i,j} - \pi_{i-m,j-m}; m) f(\pi_{i,j} - \pi_{i-m-1,j-m-1}; m)}, \quad (1.19)$$

$$f_\alpha^{\text{D}}(\pi; q, t) = \prod_{(i,i) \in \alpha} \prod_{\substack{m \geq 0 \\ m \text{ even}}} \frac{f(\pi_{i,i} - \pi_{i-m-1,i-m}; m) f(\pi_{i,i} - \pi_{i-m-2,i-m-1}; m+1)}{f(\pi_{i,i} - \pi_{i-m,i-m}; m) f(\pi_{i,i} - \pi_{i-m-2,i-m-2}; m+1)}. \quad (1.20)$$

Here we use the convention that  $\pi_{i,j} = 0$  if  $i \leq 0$  or  $j \leq 0$ . Further we use the following short notation. Let  $m$  and  $n$  be positive integers such that  $m \leq n$ . When  $\rho = (\rho_m, \dots, \rho_n)$  and  $\theta = (\theta_m, \dots, \theta_n)$  satisfy

$$0 \leq \rho_n \leq \dots \leq \rho_m \leq \theta_m \leq \dots \leq \theta_n, \quad (1.21)$$

we write

$$\Phi_m^n(\rho, \theta; q, t) = \prod_{i=m+1}^n \frac{f(\rho_{i-1} - \rho_i; 0) f(\theta_{i-1} - \rho_i; 0) f(\theta_i - \rho_{i-1}; 0) f(\theta_i - \theta_{i-1}; 0)}{f(\theta_i - \rho_i; i) f(\theta_i - \rho_i; i+1)}. \quad (1.22)$$

**Proposition 1.10.** (1) Let  $\alpha$  be a strict partition of length  $r$  and  $P = P_2(\alpha)$  the associated shifted shape. If  $\pi = (\pi_{ij})_{(i,j) \in \alpha}$  is a  $P$ -partition (1.7) satisfying the condition (1.8), then its weight  $W_P(\pi; q, t)$  is given by

$$W_P(\pi; q, t) = f_\alpha^{\text{D}}(\pi; q, t) f_\alpha^{\text{ND}}(\pi; q, t). \quad (1.23)$$

(2) Let  $\alpha$  and  $\beta$  be strict partitions of length 2. Let  $f > 0$  be a positive integer, and set  $P = P_3(\alpha, \beta; f)$  to be the bird associated with  $\alpha, \beta$  and  $f$ . If  $\pi = (\sigma, \tau; \rho, \theta)$  is a  $P$ -partition satisfying the condition (1.9), then its weight  $W_P(\pi; q, t)$  is given by

$$\begin{aligned} W_P(\pi; q, t) &= \frac{f(\sigma_{22} - \sigma_{12}; 0) f(\tau_{22} - \tau_{12}; 0) f(\rho_f; 0) f(\theta_f; f+1)}{f(\sigma_{22} - \sigma_{11}; 0) f(\sigma_{22} - \sigma_{11}; 1)} \\ &\quad \times \Phi_0^f(\rho, \theta; q, t) f_\alpha^{\text{ND}}(\sigma; q, t) f_\beta^{\text{ND}}(\tau; q, t). \end{aligned} \quad (1.24)$$

Here we use the convention that  $\sigma_{11} = \tau_{11} = \rho_0$  and  $\sigma_{22} = \tau_{22} = \theta_0$ .

(3) Let  $\alpha$  be a strict partition of length 4. Let  $P = P_6(\alpha; f)$  be the banner associated with  $\alpha$  and  $f$ . If  $\pi = (\sigma; \rho, \theta)$  is a  $P$ -partition satisfying the condition (1.10), then its weight  $W_P(\pi; q, t)$  is given by

$$W_P(\pi; q, t) = f(\rho_f; 0) f(\theta_f; f+1) \Phi_1^f(\rho, \theta; q, t) f_\alpha^{\text{D}}(\sigma; q, t) f_\alpha^{\text{ND}}(\sigma; q, t). \quad (1.25)$$

Here we use the convention that  $\sigma_{11} = \rho_1$  and  $\sigma_{33} = \theta_1$ .

**Proof.** For (1) (1.23) is exactly the same as [8, Theorem 1.2 (9)]. For (2) and (3), one can compute  $W_P(\pi; q, t)$  directly from the definition (1.17).  $\square$

**Proposition 1.11.** (1) Let  $\alpha$  be a strict partition of length  $r$  and  $P = P_2(\alpha)$  the associated shifted shape. Let  $n$  be an integer such that  $n \geq \alpha_1$ , and let  $\alpha^c$  be the strict partition formed by the complement of  $\alpha$  in  $[n]$ , i.e.,

$$\{\alpha_1, \dots, \alpha_r\} \cup \{\alpha_1^c, \dots, \alpha_{n-r}^c\} = [n].$$

We write  $y_0 = z_{0'}$  (see Proposition 1.6 (1)) hereafter. Then we have

$$F(z[H_p]; q, t) = \prod_{\alpha_i^c < \alpha_j} F(\tilde{z}_{\alpha_i^c}^{-1} \tilde{z}_{\alpha_j}; q, t) \prod_i F(\tilde{z}_{\alpha_i}; q, t) \prod_{i < j} F(w \tilde{z}_{\alpha_i} \tilde{z}_{\alpha_j}; q, t), \quad (1.26)$$

where  $\begin{cases} w = y_0/z_0 \text{ and } \tilde{z}_i = \prod_{k=0}^{i-1} z_k & (i = 1, \dots, n). & \text{if } r \text{ is odd,} \\ w = z_0/y_0 \text{ and } \tilde{z}_i = y_0 \prod_{k=1}^{i-1} z_k & (i = 1, \dots, n). & \text{if } r \text{ is even.} \end{cases}$

(2) Let  $\alpha = (\alpha_1, \alpha_2)$  and  $\beta = (\beta_1, \beta_2)$  be strict partitions of length 2. Let  $f > 0$  be a positive integer, and set  $P = P_3(\alpha, \beta; f)$  the bird associated with  $f$ ,  $\alpha$  and  $\beta$ . Let  $m, n$  be integers such that  $m \geq \ell(\alpha)$  and  $n \geq \ell(\beta)$ , and let  $\alpha^c$  (resp.  $\beta^c$ ) be the strict partition formed by the complement of  $\alpha$  (resp.  $\beta$ ) in  $[m]$  (resp.  $[n]$ ). We write  $y_i = z_{i'}$  for  $i = 1, \dots, \beta_1 - 1$  and  $x_i = z_{-i}$  for  $i = 1, \dots, f$ . Further we may write  $x_0 = y_0 = z_0$ . (See Proposition 1.6 (2)). Then we have

$$\begin{aligned} F(z[H_p]; q, t) &= \prod_{\alpha_i^c < \alpha_j} F(\tilde{z}_{\alpha_i^c}^{-1} \tilde{z}_{\alpha_j}; q, t) \prod_{\beta_i^c < \beta_j} F(\tilde{y}_{\beta_i^c}^{-1} \tilde{y}_{\beta_j}; q, t) \prod_{i=1}^f F(\tilde{x}_i; q, t) \\ &\times \prod_{i=1}^f F\left(\frac{\tilde{x}_0^2}{\tilde{x}_i} \prod_{k,l=1}^2 \tilde{y}_l \tilde{z}_k; q, t\right) \prod_{i,j=1}^2 F(\tilde{x}_0 \tilde{y}_{\beta_j} \tilde{z}_{\alpha_i}; q, t) \end{aligned} \quad (1.27)$$

where  $\tilde{x}_i = \prod_{k=i}^f x_k$  for  $i = 0, \dots, f$ ,  $\tilde{y}_i = \prod_{k=1}^{i-1} y_k$  for  $i = 1, \dots, n$ , and  $\tilde{z}_i = \prod_{k=1}^{i-1} z_k$  for  $i = 1, \dots, m$ .

(3) Let  $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$  be a strict partition of length 4. Let  $P = P_6(f; \alpha)$  the banner associated with  $\alpha$  and  $\beta$ . Let  $n$  be an integer such that  $n \geq 4 = \ell(\alpha)$ , and let  $\alpha^c$  be the strict partition formed by the complement of  $\alpha$  in  $[n]$ . We write  $y_0 = z_{0'}$  and  $x_i = z_{-i+1}$  for  $i = 2, \dots, f$  (see Proposition 1.6 (3)). Hereafter we may use the convention that  $x_1 = z_0$ . Then we have

$$\begin{aligned} F(z[H_p]; q, t) &= \prod_{\alpha_i^c < \alpha_j} F(\tilde{z}_{\alpha_i^c}^{-1} \tilde{z}_{\alpha_j}; q, t) \prod_{i=2}^f F(\tilde{x}_i; q, t) \prod_{i=2}^f F\left(\frac{\tilde{x}_2^2}{\tilde{x}_i} w^2 \prod_{i=1}^4 \tilde{z}_{\alpha_i}; q, t\right) \\ &\times \prod_{i=1}^4 F(\tilde{z}_{\alpha_i}; q, t) \prod_{1 \leq i < j \leq 4} F(\tilde{x}_2 w \tilde{z}_{\alpha_i} \tilde{z}_{\alpha_j}; q, t), \end{aligned} \quad (1.28)$$

where  $w = \frac{z_0}{y_0}$ ,  $\tilde{x}_i = \prod_{k=i}^f x_k$  for  $i = 1, \dots, f$  and  $\tilde{z}_i = y_0 \prod_{k=1}^{i-1} z_k$  for  $i = 1, \dots, n$ .

**Proof.** (1) If  $P = P_2(\alpha)$ , then we have

$$z[H_P(i, j)] = \begin{cases} \tilde{z}_{\alpha_i} \tilde{z}_{\alpha_{j+1}} & \text{if } i \leq j < r, \\ \tilde{z}_{\alpha_i} & \text{if } i \leq j = r, \\ \tilde{z}_{\alpha_{1-j+1}}^{-1} \tilde{z}_{\alpha_i} & \text{if } i \leq r < j, \end{cases}$$

as stated in [8, §3, Proof Theorem 1.2].

(2) If  $P = P_3(\alpha, \beta; f)$ , then we have

$$z[H_P(i, j)] = \begin{cases} \frac{\tilde{x}_0^2}{\tilde{x}_{-j+1}} \prod_{k,l=1}^2 \tilde{y}_k \tilde{z}_l & \text{if } i = 1 \text{ and } -f+1 \leq j \leq 0, \\ \tilde{x}_0 \tilde{y}_{\beta_j} \tilde{z}_{\alpha_i} & \text{if } 1 \leq i, j \leq 2, \\ \tilde{z}_{\alpha_1-j+1}^{-1} \tilde{z}_{\alpha_i} & \text{if } 1 \leq i \leq 2 < j, \\ \tilde{y}_{\beta_{1-i+1}}^{-1} \tilde{y}_{\beta_j} & \text{if } 1 \leq j \leq 2 < i, \\ \tilde{x}_{i-2} & \text{if } 3 \leq i = j \leq f+2. \end{cases}$$

(3) If  $P = P_6(\alpha; f)$ , then we have

$$z[H_P(i, j)] = \begin{cases} \frac{\tilde{x}_2^2}{\tilde{x}_{-j+2}} \frac{z_0^2}{y_0^2} \prod_{k=1}^4 \tilde{z}_k & \text{if } i = 1 \text{ and } -f+2 \leq j \leq 0, \\ \frac{z_0}{y_0} \tilde{x}_2 \tilde{z}_{\alpha_i} \tilde{z}_{\alpha_{j+1}} & \text{if } 1 \leq i \leq j < 4, \\ \tilde{z}_{\alpha_i} & \text{if } 1 \leq i \leq j = 4, \\ \tilde{z}_{\alpha_1-j+1}^{-1} \tilde{z}_{\alpha_i} & \text{if } 1 \leq i \leq 4 < j, \\ \tilde{x}_{i-2} & \text{if } 3 < i \leq f+2 \text{ and } j = 3. \end{cases}$$

□

## 2 Macdonald polynomials

In this section we recall the fundamental properties of Macdonald polynomials and consider its application. Especially Theorem 2.2 and its corollary will play an important role in the next section.

We follow the notation and terminology of [7] for the symmetric functions. If  $\lambda$  and  $\mu$  are partitions then  $\mu \subseteq \lambda$  if  $\mu$  is contained in  $\lambda$ , i.e.,  $\mu_i \leq \lambda_i$  for all  $i \geq 1$ . If  $\mu \subseteq \lambda$  then the skew-diagram  $\lambda/\mu$  denotes the set-theoretic difference between  $\lambda$  and  $\mu$ , i.e., those squares of  $\lambda$  not contained in  $\mu$ . The skew diagram  $\lambda/\mu$  is a vertical  $r$ -strip if  $|\lambda - \mu| = |\lambda| - |\mu| = r$  and if, for all  $i \geq 1$ ,  $\lambda_i \geq \mu_i$  is at most one, i.e., each row of  $\lambda - \mu$  contains at most one square. The set of all vertical  $r$ -strips is denoted by  $\mathcal{V}_r$  and the set of all vertical strips by  $\mathcal{V} = \biguplus_{r=0}^{\infty} \mathcal{V}_r$ . The skew diagram  $\lambda/\mu$  is a horizontal  $r$ -strip if  $|\lambda - \mu| = r$  and if, for all  $i \geq 1$ ,  $\lambda'_i - \mu'_i$  is at most one, i.e., each column of  $\lambda - \mu$  contains at most one square. For two partitions  $\lambda$  and  $\mu$ , we write  $\lambda \succ \mu$  if  $\lambda \supset \mu$  and  $\lambda/\mu$  is a horizontal strip. Note that  $\lambda/\mu$  is a horizontal strip if and only if  $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots$ . The set of all horizontal  $r$ -strips is denoted by  $\mathcal{H}_r$  and the set of all horizontal strips by  $\mathcal{H}$ . Let  $s = (i, j)$  be a square in the diagram of  $\lambda$ , and let  $a(s)$  and  $l(s)$  be the arm-length and leg-length of  $s$ , given by

$$a(s) = \lambda_i - j, \quad l(s) = \lambda'_j - i$$

Then we define the rational functions let

$$b_\lambda(s) = b_\lambda(s; q, t) := \begin{cases} \frac{1 - q^{a(s)} t^{l(s)+1}}{1 - q^{a(s)+1} t^{l(s)}}, & \text{if } s \in \lambda, \\ 1, & \text{otherwise,} \end{cases}$$

and [6, (3.6)] [7, VI.7 (6.19), VI.7 Ex.4]

$$b_\lambda(q, t) := \prod_{s \in \lambda} b_\lambda(s; q, t) = \prod_{i \geq 1} \prod_{m \geq 0} \frac{f_{q,t}(\lambda_i - \lambda_{i+m+1}; m)}{f_{q,t}(\lambda_i - \lambda_{i+m}; m)}, \quad (2.1)$$

$$b_\lambda^{\text{el}}(q, t) := \prod_{\substack{s \in \lambda \\ l(s) \text{ even}}} b_\lambda(s; q, t) = \prod_{i \geq 1} \prod_{\substack{m \geq 0 \\ m \text{ even}}} \frac{f_{q,t}(\lambda_i - \lambda_{i+m+1}; m)}{f_{q,t}(\lambda_i - \lambda_{i+m}; m)}, \quad (2.2)$$

$$b_\lambda^{\text{oa}}(q, t) := \prod_{\substack{s \in \lambda \\ a(s) \text{ odd}}} b_\lambda(s; q, t). \quad (2.3)$$

If  $x = (x_1, x_2, \dots)$  and  $y = (y_1, y_2, \dots)$  are two sequences of independent indeterminates, then we write

$$\Pi(x; y; q, t) = \prod_{i,j} \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty} = \prod_{i,j} F(x_i y_j; q, t). \quad (2.4)$$

Let  $\mathfrak{S}_n$  denote the symmetric group, acting on  $x = (x_1, \dots, x_n)$  by permuting the  $x_i$ , and let  $\Lambda_n = \mathbb{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n}$  and  $\Lambda$  denote the ring of symmetric polynomials in  $n$  independent variables and the ring of symmetric polynomials in countably many variables, respectively. For  $\lambda = (\lambda_1, \dots, \lambda_n)$  a partition of at most  $n$  parts the monomial symmetric function  $m_\lambda$  is defined as

$$m_\lambda(x) = \sum_{\alpha} x^\alpha$$

where the sum is over all distinct permutations  $\alpha$  of  $\lambda$ , and  $x = (x_1, \dots, x_n)$ . For  $\ell(\lambda) > n$  we set  $m_\lambda(x) = 0$ . The monomial symmetric functions  $m_\lambda(x)$  for  $\ell(\lambda) \leq n$  form a  $\mathbb{Z}$ -basis of  $\Lambda_n$ . For  $r$  a nonnegative integer the power sums  $p_r$  are given by  $p_0 = 1$  and  $p_r = m_{(r)}$  for  $r > 1$ . More generally the power-sum products are defined as  $p_\lambda(x) = p_{\lambda_1}(x)p_{\lambda_2}(x) \cdots$  for an arbitrary partition  $\lambda = (\lambda_1, \lambda_2, \dots)$ . Define the Macdonald scalar product  $\langle \cdot, \cdot \rangle_{q,t}$  on the ring of symmetric functions by

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_i \prod_{i=1}^n \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}$$

with  $z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!$  and  $m_i = m_i(\lambda)$ . If we denote the ring of symmetric functions in  $\Lambda_n$  variables over the field  $\mathbb{F} = \mathbb{Q}(q, t)$  of rational functions in  $q$  and  $t$  by  $\Lambda_{n,\mathbb{F}}$ , then the Macdonald polynomial  $P_\lambda(x) = P_\lambda(x; q, t)$  is the unique symmetric polynomial in  $\Lambda_{n,\mathbb{F}}$  such that [VI (4.7)]Mac:

$$P_\lambda = \sum_{\mu \leq \lambda} u_{\lambda\mu}(q, t) m_\mu(x)$$

with  $u_{\lambda\lambda} = 1$  and

$$\langle P_\lambda, P_\mu \rangle_{q,t} = 0 \quad \text{if } \lambda \neq \mu.$$

The Macdonald polynomials  $P_\lambda(x; q, t)$  with  $\ell(\lambda) \leq n$  form an  $\mathbb{F}$ -basis of  $\Lambda_{n,\mathbb{F}}$ . If  $\ell(\lambda) > n$  then  $P_\lambda(x; q, t) = 0$ .  $P_\lambda(x; q, t)$  is called *Macdonald's P-function*. Since  $P_\lambda(x_1, \dots, x_n, 0; q, t) = P_\lambda(x_1, \dots, x_n; q, t)$  one can extend the Macdonald polynomials to symmetric functions containing an infinite number of independent variables  $x = (x_1, x_2, \dots)$ , to obtain a basis of  $\mathbb{F} = \Lambda \otimes \mathbb{F}$ . A second Macdonald symmetric function, called *Macdonald's Q-function*, is defined as

$$Q_\lambda(x; q, t) = b_\lambda(q, t) P_\lambda(x; q, t). \quad (2.5)$$

The normalization of the Macdonald inner product is then  $\langle P_\lambda, Q_\mu \rangle_{q,t} = \delta_{\lambda\mu}$  for all  $\lambda, \mu$ , which is equivalent to

$$\sum_{\lambda} P_\lambda(x; q, t) Q_\lambda(y; q, t) = \Pi(x; y; q, t). \quad (2.6)$$

(See [7, VI.4, (4.13)].) Let  $g_r(x; q, t) := Q_{(r)}(x; q, t)$ , or equivalently, [7, VI.2, (2.8)]

$$\prod_{i=1}^{\infty} \frac{(tx_i y; q)_\infty}{(x_i y; q)_\infty} = \sum_{r=0}^{\infty} g_r(x; q, t) y^r.$$

Then the *Pieri coefficients*  $\phi_{\lambda/\mu}$  and  $\psi_{\lambda/\mu}$  are given by [7, VI.6, (6.24)]

$$\begin{aligned} P_\mu(x; q, t) g_r(x; q, t) &= \sum_{\substack{\lambda \\ \lambda - \mu \in \mathcal{H}_r}} \phi_{\lambda/\mu}(q, t) P_\lambda(x; q, t), \\ Q_\mu(x; q, t) g_r(x; q, t) &= \sum_{\substack{\lambda \\ \lambda - \mu \in \mathcal{H}_r}} \psi_{\lambda/\mu}(q, t) Q_\lambda(x; q, t). \end{aligned}$$

Another direct expressions for  $\phi_{\lambda/\mu}$  and  $\psi_{\lambda/\mu}$  is given in [7, VI.6, Ex.2] as

$$\phi_{\lambda/\mu}(q, t) = \prod_{1 \leq i \leq j \leq \ell(\lambda)} \frac{f(\lambda_i - \mu_j; j - i) f(\mu_i - \lambda_{j+1}; j - i)}{f(\lambda_i - \lambda_j; j - i) f(\mu_i - \mu_{j+1}; j - i)}, \quad (2.7)$$

$$\psi_{\lambda/\mu}(q, t) = \prod_{1 \leq i \leq j \leq \ell(\mu)} \frac{f(\lambda_i - \mu_j; j - i) f(\mu_i - \lambda_{j+1}; j - i)}{f(\mu_i - \mu_j; j - i) f(\lambda_i - \lambda_{j+1}; j - i)}. \quad (2.8)$$

Here we use these expressions to rewrite Okada's  $(q, t)$ -hook formula conjectures by the Pieri coefficients. For any three partitions  $\lambda, \mu, \nu$  let  $f_{\mu\nu}^\lambda$  be the coefficient  $P_\lambda$  in the product  $P_\mu P_\nu$ : [7, VI (7.1')]:

$$P_\mu(x; q, t) P_\nu(x; q, t) = \sum_\lambda f_{\mu\nu}^\lambda P_\lambda(x; q, t) \quad (2.9)$$

Now let  $\lambda, \mu$  be partitions and define  $Q_{\lambda/\mu} \in \Lambda_{\mathbb{F}}$  by

$$Q_{\lambda/\mu}(x; q, t) = \sum_\nu f_{\mu\nu}^\lambda Q_\nu(x; q, t). \quad (2.10)$$

Then  $Q_{\lambda/\mu}(x; q, t) = 0$  unless  $\lambda \supset \mu$ , and  $Q_{\lambda/\mu}$  is homogeneous of degree  $|\lambda| - |\mu|$ , which is called *Macdonald's skew  $Q$ -function*. We define *Macdonald's skew  $P$ -function*  $P_{\lambda/\mu}$  as

$$Q_{\lambda/\mu}(x; q, t) = \frac{b_\lambda(q, t)}{b_\mu(q, t)} P_{\lambda/\mu}(x; q, t). \quad (2.11)$$

holds. Let  $T$  be a tableau of shape  $\lambda - \mu$  and weight  $\nu$ , thought as a sequence of partitions  $(\lambda^{(0)}, \dots, \lambda^{(r)})$  such that

$$\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \dots \subset \lambda^{(r)} = \lambda$$

and such that each  $\lambda^{(i)} - \lambda^{(i-1)}$  is a horizontal strip. Let

$$\begin{aligned} \phi_T(q, t) &= \prod_{i=1}^r \phi_{\lambda^{(i)}/\lambda^{(i-1)}}(q, t), \\ \psi_T(q, t) &= \prod_{i=1}^r \psi_{\lambda^{(i)}/\lambda^{(i-1)}}(q, t). \end{aligned}$$

Then we have [7, VI, (7.13), (7.13')]

$$\begin{aligned} Q_{\lambda/\mu}(x; q, t) &= \sum_T \phi_T(q, t) x^T, \\ P_{\lambda/\mu}(x; q, t) &= \sum_T \psi_T(q, t) x^T, \end{aligned}$$

summed over tableaux  $T$  of shape  $\lambda - \mu$ , where  $x^T = \prod_{i=1}^r x_i^{|\lambda^{(i)} - \lambda^{(i-1)}|}$ . It also holds [7, VI.7, (7.9) (7.9')]

$$Q_\lambda(x, z; q, t) = \sum_\mu Q_{\lambda/\mu}(x, z; q, t) Q_\mu(x, z; q, t), \quad (2.12)$$

$$P_\lambda(x, z; q, t) = \sum_\mu P_{\lambda/\mu}(x, z; q, t) P_\mu(x, z; q, t), \quad (2.13)$$

where the sums on the right are over partitions  $\mu \subset \lambda$ . The following lemma has appeared in the proof of [13, Proposition 2.2] (also see [7, I.5, Ex.26] and [14, Proposition 5.1]).

**Lemma 2.1.** Let  $\mu$  and  $\nu$  be partitions, and  $x = (x_1, x_2, \dots)$  and  $y = (y_1, y_2, \dots)$  are independent indeterminates.

$$\sum_\lambda Q_{\lambda/\mu}(x; q, t) P_{\lambda/\nu}(y; q, t) = \Pi(x; y; q, t) \sum_\tau Q_{\nu/\tau}(x; q, t) P_{\mu/\tau}(y; q, t) \quad (2.14)$$

**Proof.** First if we use (2.12) (2.13) and (2.6), then we have

$$\begin{aligned}
& \sum_{\mu, \nu} \sum_{\lambda} Q_{\lambda/\mu}(x) P_{\lambda/\nu}(y) Q_{\mu}(z) P_{\nu}(w) \\
&= \sum_{\mu, \nu} \sum_{\lambda} Q_{\lambda}(x, z) P_{\lambda}(y, w) \\
&= \Pi(x, z; y, w) \\
&= \Pi(x; y) \Pi(x; w) \Pi(z; y) \Pi(z; w) \\
&= \Pi(x; y) \sum_{\xi} Q_{\xi}(x) P_{\xi}(w) \sum_{\eta} Q_{\eta}(z) P_{\eta}(y) \sum_{\tau} Q_{\tau}(z) P_{\tau}(w)
\end{aligned}$$

by (2.9) and (2.5)

$$= \Pi(x; y) \sum_{\xi, \eta, \tau} Q_{\xi}(x) P_{\eta}(y) \sum_{\mu} \frac{b_{\eta} b_{\tau}}{b_{\mu}} f_{\eta\tau}^{\mu} Q_{\mu}(z) \sum_{\nu} f_{\xi\tau}^{\nu} P_{\nu}(w)$$

by (2.10) and (2.11)

$$= \Pi(x; y) \sum_{\mu, \nu, \tau} Q_{\nu/\tau}(x) P_{\mu/\tau}(y) Q_{\mu}(z) P_{\nu}(w).$$

Hence, by comparing the coefficients of  $Q_{\mu}(z) P_{\nu}(w)$  in the both sides, we obtain the desired identity. This completes the proof.  $\square$

In [13] Vuletić has presented so-called a generalized MacMahon's formula. The following theorem gives a generalized form of [13, Proposition 2.2], which we use in the proof of Okada's conjecture.

**Theorem 2.2.** Fix a positive integer  $T$  and two partitions  $\mu^0$  and  $\mu^T$ . Let  $x^0, \dots, x^{T-1}, y^1, \dots, y^T$  be sets of variables. Then we have

$$\begin{aligned}
& \sum_{(\lambda^1, \mu^1, \lambda^2, \dots, \lambda^T)} \prod_{i=1}^T Q_{\lambda^i/\mu^{i-1}}(x^{i-1}; q, t) P_{\lambda^i/\mu^i}(y^i; q, t) \\
&= \prod_{0 \leq i < j \leq T} \Pi(x^i; y^j; q, t) \sum_{\nu} Q_{\mu^T/\nu}(x^0, \dots, x^{T-1}; q, t) P_{\mu^0/\nu}(y^1, \dots, y^T; q, t)
\end{aligned} \tag{2.15}$$

where the sum runs over  $(2T-1)$ -tuples  $(\lambda^1, \mu^1, \lambda^2, \dots, \mu^{T-1}, \lambda^T)$  of partitions satisfying

$$\mu^0 \subset \lambda^1 \supset \mu^1 \subset \lambda^2 \supset \mu^2 \subset \dots \supset \mu^{T-1} \subset \lambda^T \supset \mu^T. \tag{2.16}$$

**Proof.** Our proof is similar to that of [13, Proposition 2.2]. We proceed by induction on  $T$ . If  $T = 1$  then Lemma 2.1 is nothing but the desired identity. Assume  $T > 1$  and (2.15) holds up to  $T-1$ . We need consider the sum

$$S := \sum_{(\lambda^1, \mu^1, \lambda^2, \dots, \lambda^T)} \prod_{i=1}^T Q_{\lambda^i/\mu^{i-1}}(x^{i-1}) P_{\lambda^i/\mu^i}(y^i),$$

where the sum runs over  $(\lambda^1, \mu^1, \lambda^2, \dots, \lambda^T)$  satisfying (2.16). First fix  $(\mu^1, \dots, \mu^{T-1})$  and take the sum over  $(\lambda^1, \dots, \lambda^T)$  using (2.14). Then we obtain

$$S = \prod_{i=1}^T \Pi(x^{i-1}; y^i) \sum_{(\tau^1, \mu^1, \dots, \tau^T)} \prod_{i=1}^T Q_{\mu^i/\tau^i}(x^{i-1}) P_{\mu^{i-1}/\tau^i}(y^i),$$

where the sum runs over  $(\lambda^1, \mu^1, \lambda^2, \dots, \lambda^T)$  satisfying

$$\mu^0 \supset \tau^1 \subset \mu^1 \supset \tau^2 \subset \mu^2 \supset \dots \supset \mu^{T-1} \supset \tau^T \subset \mu^T.$$

By the induction hypothesis we can suppose

$$\begin{aligned} & \sum_{(\mu^1, \tau^2, \dots, \mu^{T-1})} \sum_{i=1}^T \prod Q_{\mu^i / \tau^i}(x^{i-1}) P_{\mu^{i-1} / \tau^i}(y^i) \\ &= \prod_{0 \leq i < j \leq T-1} \Pi(x^i; y^{j+1}) \sum_{\nu} Q_{\tau^T / \nu}(x^0, \dots, x^{T-2}) P_{\tau^1 / \nu}(y^2, \dots, y^T). \end{aligned}$$

Hence, substituting this identity into the above  $S$ , we obtain

$$S = \prod_{0 \leq i < j \leq T} \Pi(x^i; y^j) \sum_{(\tau^1, \nu, \tau^T)} Q_{\mu^T / \tau^T}(x^{T-1}) P_{\mu^0 / \tau^1}(y^1) Q_{\tau^T / \nu}(x^0, \dots, x^{T-2}) P_{\tau^1 / \nu}(y^2, \dots, y^T),$$

where the sum runs over  $(\tau^1, \nu, \tau^T)$  such that

$$\mu^0 \supset \tau^1 \supset \nu \subset \tau^T \subset \mu^T.$$

Applying (2.12) and (2.13), we obtain the desired identity for  $T$ . This completes the theorem.  $\square$

We define  $P_{[\lambda, \mu]}^\delta(x; q, t)$  and  $Q_{[\lambda, \mu]}^\delta(x; q, t)$  for a pair  $(\lambda, \mu)$  of partitions, a set  $x = (x_1, x_2, \dots)$  of independent variables and  $\delta = \pm 1$  by

$$P_{[\lambda, \mu]}^\delta(x; q, t) = \begin{cases} P_{\lambda / \mu}(x; q, t) & \text{if } \delta = +1, \\ Q_{\mu / \lambda}(x; q, t) & \text{if } \delta = -1, \end{cases} \quad Q_{[\lambda, \mu]}^\delta(q, t) = \begin{cases} Q_{\lambda / \mu}(x; q, t) & \text{if } \delta = +1, \\ P_{\mu / \lambda}(x; q, t) & \text{if } \delta = -1. \end{cases}$$

Here we assume  $\lambda \supset \mu$  if  $\delta = +1$ , and  $\lambda \subset \mu$  if  $\delta = -1$ .

**Corollary 2.3.** Let  $n$  be a positive integer, and  $\epsilon = (\epsilon_1, \dots, \epsilon_n)$  a sequence of  $\pm 1$ . Fix a positive integer  $T$  and two partitions  $\lambda^0$  and  $\lambda^n$ . Let  $x^1, \dots, x^n$  be sets of variables. Then we have

$$\begin{aligned} & \sum_{(\lambda^1, \lambda^2, \dots, \lambda^{n-1})} \prod_{i=1}^n P_{[\lambda^{i-1}, \lambda^i]}^{\epsilon_i}(x^i; q, t) \\ &= \prod_{\substack{i < j \\ (\epsilon_i, \epsilon_j) = (-1, +1)}} \Pi(x^i; x^j; q, t) \sum_{\nu} Q_{\lambda^n / \nu}(\{x^i\}_{\epsilon_i = -1}; q, t) P_{\lambda^0 / \nu}(\{x^i\}_{\epsilon_i = +1}; q, t), \end{aligned} \quad (2.17)$$

$$\begin{aligned} & \sum_{(\lambda^1, \lambda^2, \dots, \lambda^{n-1})} \prod_{i=1}^n Q_{[\lambda^{i-1}, \lambda^i]}^{\epsilon_i}(x^i; q, t) \\ &= \prod_{\substack{i < j \\ (\epsilon_i, \epsilon_j) = (-1, +1)}} \Pi(x^i; x^j; q, t) \sum_{\nu} P_{\lambda^n / \nu}(\{x^i\}_{\epsilon_i = -1}; q, t) Q_{\lambda^0 / \nu}(\{x^i\}_{\epsilon_i = +1}; q, t), \end{aligned} \quad (2.18)$$

where the sum runs over  $(n-1)$ -tuples  $(\lambda^1, \lambda^2, \dots, \lambda^{n-1})$  of partitions satisfying

$$\begin{cases} \lambda^{i-1} \supset \lambda^i & \text{if } \epsilon_i = +1, \\ \lambda^{i-1} \subset \lambda^i & \text{if } \epsilon_i = -1. \end{cases} \quad (2.19)$$

**Proof.** Taking  $T = n$  in Theorem 2.2, we have

$$\begin{aligned} & \sum_{(\Lambda^1, \Lambda^1, \Lambda^2, \dots, \Lambda^n)} \prod_{i=1}^n Q_{\Lambda^i / \Lambda^{i-1}}(X^{i-1}; q, t) P_{\Lambda^i / \Lambda^i}(Y^i; q, t) \\ &= \prod_{0 \leq i < j \leq n} \Pi(X^i; Y^j; q, t) \sum_{\nu} Q_{\lambda^n / \nu}(X^0, \dots, X^{n-1}; q, t) P_{\lambda^0 / \nu}(Y^1, \dots, Y^n; q, t) \end{aligned}$$

where the sum runs over

$$\lambda^0 \subset \Lambda^1 \supset \lambda^1 \subset \Lambda^2 \supset \lambda^2 \subset \dots \subset \lambda^{n-1} \subset \Lambda^n \supset \lambda^n.$$

Put  $X^{i-1} = 0$  and  $Y^i = x^i$  if  $\epsilon_i = +1$ , and  $X^{i-1} = x^i$  and  $Y^i = 0$  if  $\epsilon_i = -1$ . Since  $P_{\lambda/\mu}(0; q, t) = Q_{\lambda/\mu}(0; q, t) = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{otherwise,} \end{cases}$  we obtain  $\Lambda^i = \begin{cases} \lambda^{i-1} & \text{if } \epsilon_i = +1, \\ \lambda^i & \text{if } \epsilon_i = -1. \end{cases}$  Hence we have the condition (2.19) for the sum. Since

$$Q_{\Lambda^i/\Lambda^{i-1}}(X^{i-1}; q, t) P_{\Lambda^i/\Lambda^i}(Y^i; q, t) = \begin{cases} P_{\lambda^{i-1}/\lambda^i}(x^i; q, t) & \text{if } \epsilon_i = +1, \\ Q_{\lambda^i/\lambda^{i-1}}(x^i; q, t) & \text{if } \epsilon_i = -1. \end{cases}$$

the left-hand side equals

$$\sum_{(\lambda^1, \lambda^2, \dots, \lambda^{n-1})} \prod_{i=1}^n P_{[\lambda^{i-1}, \lambda^i]}^{\epsilon_i}(x^i; q, t).$$

Meanwhile the right-hand side becomes

$$\prod_{\substack{i < j \\ (\epsilon_i, \epsilon_j) = (-1, +1)}} \Pi(x^i; x^j; q, t) \sum_{\nu} Q_{\lambda^n/\nu}(\{x^i\}_{\epsilon_i = -1}; q, t) P_{\lambda^0/\nu}(\{x^i\}_{\epsilon_i = +1}; q, t).$$

This proves (2.17). The other identity can be proven similarly.  $\square$

**Theorem 2.4.** (Warnaar [15, Proposition 1.3, (1.17)])

$$\sum_{\lambda} w^{r(\lambda)} b_{\lambda}^{\text{oa}}(q, t) P_{\lambda}(x; q, t) = \prod_{i \geq 1} \frac{(1 + wx_i)(qtx_i^2; q^2)_{\infty}}{(x_i^2; q^2)_{\infty}} \prod_{i < j} \frac{(tx_i x_j; q)_{\infty}}{(x_i x_j; q)_{\infty}}, \quad (2.20)$$

where  $r(\lambda)$  is the number of rows of odd length.

Applying  $w_{q,t}$  [7, VI.2, (2.14)] to the both sides of (2.20), we obtain

**Corollary 2.5.**

$$\sum_{\lambda} w^{r(\lambda')} b_{\lambda}^{\text{el}}(q, t) P_{\lambda}(x; q, t) = \prod_{i \geq 1} \frac{(twx_i; q)_{\infty}}{(wx_i; q)_{\infty}} \prod_{i < j} \frac{(tx_i x_j; q)_{\infty}}{(x_i x_j; q)_{\infty}}. \quad (2.21)$$

**Proof.** First, if we take logarithm of the right-hand side of (2.20), then we have

$$\begin{aligned} \log \prod_{i < j} \frac{(tx_i x_j; q)_{\infty}}{(x_i x_j; q)_{\infty}} &= \sum_{i < j} \sum_{r \geq 0} \{ \log(1 - q^r tx_i x_j) - \log(1 - q^r x_i x_j) \} \\ &= \sum_{n \geq 1} \frac{1}{n} \cdot \frac{1 - t^n}{1 - q^n} \sum_{i < j} x_i^n x_j^n \\ &= \frac{1}{2} \sum_{n \geq 1} \frac{1}{n} \cdot \frac{1 - t^n}{1 - q^n} \{ p_n(x)^2 - p_{2n}(x) \}. \end{aligned}$$

Applying the  $\mathbb{F}$ -algebra homomorphism  $w_{q,t}$  to this formula, and using  $w_{q,t} p_r(x) = (-1)^{r-1} \frac{1-q^r}{1-t^r} p_r(x)$  (see [7, VI.2, (2.14)]), we obtain

$$w_{q,t} \log \prod_{i < j} \frac{(tx_i x_j; q)_{\infty}}{(x_i x_j; q)_{\infty}} = \sum_{n \geq 1} \frac{1}{n} \cdot \frac{1 - q^n t^n}{1 - t^{2n}} \sum_i x_i^{2n} + \sum_{n \geq 1} \frac{1}{n} \cdot \frac{1 - q^n}{1 - t^n} \sum_{i < j} x_i^n x_j^n.$$

Similarly, since

$$\log \prod_{i \geq 1} \frac{(qtx_i^2; q^2)_{\infty}}{(x_i^2; q^2)_{\infty}} = \sum_{n \geq 1} \frac{1}{n} \cdot \frac{1 - q^n t^n}{1 - q^{2n}} p_{2n}(x),$$



we have

$$w_{q,t} \log \prod_{i \geq 1} \frac{(qtx_i^2; q^2)_\infty}{(x_i^2; q^2)_\infty} = - \sum_{n \geq 1} \frac{1}{n} \cdot \frac{1 - q^n t^n}{1 - t^{2n}} \sum_i x_i^{2n}.$$

Finally, from

$$\log \prod_i (1 + wx_i) = \sum_{n \geq 1} \frac{(-1)^{n-1} w^n}{n} p_n(x)$$

we obtain

$$w_{q,t} \log \prod_i (1 + wx_i) = \sum_i \sum_{n \geq 1} \frac{1}{n} \cdot \frac{1 - q^n}{1 - t^n} w^n x_i^n.$$

Hence we obtain

$$w_{q,t} \prod_{i \geq 1} \frac{(1 + wx_i) (qtx_i^2; q^2)_\infty}{(x_i^2; q^2)_\infty} \prod_{i < j} \frac{(tx_i x_j; q)}{(x_i x_j; q)} = \prod_i \frac{(qwx_i; t)_\infty}{(wx_i; t)_\infty} \prod_{i < j} \frac{(qx_i x_j; t)_\infty}{(x_i x_j; t)_\infty}.$$

Now, applying

$$\begin{aligned} w_{q,t} P_\lambda(x; q, t) &= Q_{\lambda'}(x; t, q) \\ w_{q,t} Q_\lambda(x; q, t) &= P_{\lambda'}(x; t, q) \end{aligned}$$

([7, VI.5, (5.1)]) to the left-hand side of (2.20), and swapping  $q$  and  $t$ , we obtain the desired formula (2.21).  $\square$

From (2.21), we easily obtain

$$\sum_{\lambda} w^{\frac{|\lambda| + r(\lambda')}{2}} b_{\lambda}^{\text{el}}(q, t) P_{\lambda}(x; q, t) = \prod_{i \geq 1} \frac{(twx_i; q)_{\infty}}{(wx_i; q)_{\infty}} \prod_{i < j} \frac{(twx_i x_j; q)_{\infty}}{(wx_i x_j; q)_{\infty}}, \quad (2.22)$$

and

$$\sum_{\lambda} w^{\frac{|\lambda| - r(\lambda')}{2}} b_{\lambda}^{\text{el}}(q, t) P_{\lambda}(x; q, t) = \prod_{i \geq 1} \frac{(tx_i; q)_{\infty}}{(x_i; q)_{\infty}} \prod_{i < j} \frac{(twx_i x_j; q)_{\infty}}{(wx_i x_j; q)_{\infty}}. \quad (2.23)$$

### 3 $(q, t)$ -hook formula and Macdonald polynomials

In this section we rewrite the left-hand side and the right-hand side of Okada's conjecture using the Macdonald polynomials. In Proposition 1.10 we give the left-hand sides for birds and banners, and in Proposition 1.11 we give the right-hand sides. We rewrite these formula into Theorem 3.2 and Theorem 3.3. We will see Corollary 2.3 and Corollary 2.5 plays a key role in the proof.

We define  $\phi_{[\lambda, \mu]}^{\delta}(q, t)$  and  $\psi_{[\lambda, \mu]}^{\delta}(q, t)$  for a pair  $(\lambda, \mu)$  of partitions and  $\delta = \pm 1$  by

$$\phi_{[\lambda, \mu]}^{\delta}(q, t) = \begin{cases} \phi_{\lambda/\mu}(q, t) & \text{if } \delta = +1, \\ \psi_{\mu/\lambda}(q, t) & \text{if } \delta = -1, \end{cases} \quad \psi_{[\lambda, \mu]}^{\delta}(q, t) = \begin{cases} \psi_{\lambda/\mu}(q, t) & \text{if } \delta = +1, \\ \phi_{\mu/\lambda}(q, t) & \text{if } \delta = -1. \end{cases}$$

Here we assume  $\lambda \succ \mu$  if  $\delta = +1$ , and  $\lambda \prec \mu$  if  $\delta = -1$ . We also write

$$|\lambda - \mu|_{\delta} = \begin{cases} |\lambda - \mu| & \text{if } \delta = +1, \\ |\mu - \lambda| & \text{if } \delta = -1. \end{cases}$$

Let  $n$  be a positive integer. Let  $\epsilon = (\epsilon_1, \dots, \epsilon_n)$  be a sequence of  $\pm 1$ . Let  $(\lambda^0, \lambda^1, \dots, \lambda^n)$  be an  $(n+1)$ -tuple of partitions such that  $\lambda^{i-1} \succ \lambda^i$  if  $\epsilon_i = +1$ , and  $\lambda^{i-1} \prec \lambda^i$  if  $\epsilon_i = -1$ . Then we write

$$\phi_{[\lambda^0, \lambda^1, \dots, \lambda^n]}^{\epsilon}(q, t) = \prod_{i=1}^n \phi_{[\lambda^{i-1}, \lambda^i]}^{\epsilon_i}(q, t), \quad \psi_{[\lambda^0, \lambda^1, \dots, \lambda^n]}^{\epsilon}(q, t) = \prod_{i=1}^n \psi_{[\lambda^{i-1}, \lambda^i]}^{\epsilon_i}(q, t).$$

Let  $\alpha$  be a strict partition, and let  $n$  be an integer such that  $n \geq \alpha_1$ . Define a sequence  $\epsilon = \epsilon_n(\alpha) = (\epsilon_1, \dots, \epsilon_n)$  of  $\pm 1$  by putting

$$\epsilon_k(\alpha) = \begin{cases} +1 & \text{if } k \text{ is a part of } \alpha, \\ -1 & \text{if } k \text{ is not a part of } \alpha. \end{cases}$$

For example, if  $\alpha = (8, 5, 2, 1)$  and  $n = 10$ , then we have  $\epsilon = (+ + - - + - - + - -)$ . Let  $\pi \in \mathcal{A}(P)$  a  $P$ -partition for the shifted shape  $P = P_2(\alpha)$ . For each integer  $k = 0, \dots, n$  we define the  $k$ th trace  $\pi[k]$  to be the sequence  $(\dots, \pi_{2,k+2}, \pi_{1,k+1})$  obtained by reading the  $k$ th diagonal from SE to NW. Here we use the convention that  $\pi[k] = \emptyset$  if  $k \geq \alpha_1$ . For example, if  $\pi$  is the  $P$ -partition of shifted shape  $\alpha = (8, 5, 2, 1)$  in Figure 3, then we have  $\pi[0] = (\pi_{44}, \pi_{33}, \pi_{22}, \pi_{11})$ ,  $\pi[1] = (\pi_{34}, \pi_{23}, \pi_{12})$ ,  $\pi[2] = (\pi_{24}, \pi_{13})$ ,  $\pi[3] = (\pi_{25}, \pi_{14})$ ,  $\pi[4] = (\pi_{26}, \pi_{15})$ ,  $\pi[5] = (\pi_{16})$ ,  $\pi[6] = (\pi_{17})$ ,  $\pi[7] = (\pi_{18})$ ,  $\pi[8] = \pi[9] = \pi[10] = \emptyset$ , and

$$\pi[0] \succ \pi[1] \succ \pi[2] \prec \pi[3] \prec \pi[4] \succ \pi[5] \prec \pi[6] \prec \pi[7] \succ \pi[8] \prec \pi[9] \prec \pi[10].$$

By direct computation one can easily check

$$\begin{aligned} W_P(\pi; q, t) &= b_{\pi[0]}^{\text{el}}(q, t) \psi_{[\pi[0], \dots, \pi[10]]}^{\epsilon(\alpha)}(q, t) = b_{\pi[0]}^{\text{el}} \psi_{\pi[0]/\pi[1]} \psi_{\pi[1]/\pi[2]} \phi_{\pi[3]/\pi[2]} \\ &\quad \times \phi_{\pi[4]/\pi[3]} \psi_{\pi[4]/\pi[5]} \phi_{\pi[6]/\pi[5]} \phi_{\pi[7]/\pi[6]} \psi_{\pi[7]/\pi[8]} \phi_{\pi[9]/\pi[8]} \phi_{\pi[10]/\pi[9]}. \end{aligned}$$

In the following we write

$$\begin{aligned} \widehat{\Phi}_m^n(\rho, \theta; q, t) &= \frac{f(\rho_n; 0) f(\theta_n; n+1)}{f(\rho_m; 0) (\theta_m; m+1)} \Phi_m^n(\rho, \theta; q, t), \\ \widetilde{\Phi}_m^n(\widetilde{x}; \rho, \theta; q, t) &= \widehat{\Phi}_m^n(\rho, \theta; q, t) \prod_{i=m+1}^n \widetilde{x}_i^{\rho_i + \theta_i - \rho_{i-1} - \theta_{i-1}} \end{aligned}$$

in short, where  $\rho = (\rho_m, \dots, \rho_n)$  and  $\theta = (\theta_m, \dots, \theta_n)$  satisfy (1.21), and  $\widetilde{x} = (\widetilde{x}_m, \dots, \widetilde{x}_n)$  are indeterminates. For example, if  $\pi = (\sigma, \tau; f)$  is the  $P$ -partition of the bird  $P = P_3(\alpha, \beta; f)$  for  $\alpha = (4, 3)$ ,  $\beta = (4, 2)$  and  $f = 2$  (see Figure 7) and satisfies (1.9), then we have

$$W_P(\pi; q, t) = \widehat{\Phi}_0^2(\rho, \theta; q, t) \psi_{[\sigma[0], \dots, \sigma[4]]}^{\epsilon(\alpha)}(q, t) \phi_{[\tau[0], \dots, \tau[4]]}^{\epsilon(\beta)}(q, t).$$

Further, if  $\pi = (\sigma; f)$  is the  $P$ -partition of the banner  $P = P_5(\alpha; f)$  for  $\alpha = (10, 6, 3, 2)$  and  $f = 2$  (see Figure 7) and satisfies (1.10), then we have

$$W_P(\pi; q, t) = \widehat{\Phi}_1^2(\rho, \theta; q, t) b_{\sigma[0]}^{\text{el}}(q, t) \psi_{[\sigma[0], \dots, \sigma[10]]}^{\epsilon(\alpha)}(q, t).$$

**Proposition 3.1.** (1) Let  $P = P_2(\alpha)$  be the shifted shape associated with a strict partition  $\alpha$  such that  $\ell(\alpha) = r$ , and let  $n$  be an integer such that  $n \geq \alpha_1$ . If  $\pi \in \mathcal{A}(P)$  is a  $P$ -partition satisfying the condition (1.8), then we have

$$W_P(\pi; q, t) = b_{\pi[0]}^{\text{el}}(q, t) \psi_{[\pi[0], \dots, \pi[n]]}^{\epsilon(\alpha)}(q, t) = \frac{b_{\pi[0]}^{\text{el}}(q, t)}{b_{\pi[0]}(q, t)} \phi_{[\pi[0], \dots, \pi[n]]}^{\epsilon(\alpha)}(q, t) \quad (3.1)$$

and

$$z^\pi = w^{\frac{|\pi[0]| - r(\pi[0]')}{2}} \prod_{i=1}^n \widetilde{z}_i^{\epsilon_i(\alpha) |\pi[i-1] - \pi[i]|_{\epsilon_i(\alpha)}}, \quad (3.2)$$

where  $w$  and  $\widetilde{z}_i$  ( $1 \leq i \leq n$ ) are as in Proposition 1.11 (1).

(2) Let  $\alpha = (\alpha_1, \alpha_2)$  and  $\beta = (\beta_1, \beta_2)$  be strict partitions such that  $\ell(\alpha) = \ell(\beta) = 2$ . Let  $f > 0$  be a positive integer, and set  $P = P_3(\alpha, \beta; f)$  the bird associated with  $\alpha$ ,  $\beta$  and  $f$ . Let  $m$

(resp.  $n$ ) be a positive integer such that  $m \geq \alpha_1$  (resp.  $n \geq \beta_1$ ). If  $\pi = (\sigma, \tau; \rho, \theta)$  is a  $P$ -partition satisfying the condition (1.9), then we have

$$W_P(\pi; q, t) = \widehat{\Phi}_0^f(\rho, \theta; q, t) \psi_{[\sigma[0], \dots, \sigma[m]]}^{\epsilon(\alpha)}(q, t) \phi_{[\tau[0], \dots, \tau[n]]}^{\epsilon(\beta)}(q, t) \quad (3.3)$$

and

$$z^\pi = \widetilde{x}_0^{\rho_0 + \theta_0} \prod_{i=1}^m \widetilde{z}_i^{\epsilon_i(\alpha) |\sigma[i-1] - \sigma[i]|_{\epsilon_i(\alpha)}} \prod_{i=1}^n \widetilde{y}_i^{\epsilon_i(\beta) |\tau[i-1] - \tau[i]|_{\epsilon_i(\beta)}} \prod_{i=1}^f \widetilde{x}_i^{\rho_i + \theta_i - \rho_{i-1} - \theta_{i-1}}, \quad (3.4)$$

where  $\widetilde{x}_i$  ( $0 \leq i \leq f$ ),  $\widetilde{y}_i$  ( $1 \leq i \leq n$ ) and  $\widetilde{z}_i$  ( $1 \leq i \leq m$ ) are as in Proposition 1.11 (2).

- (3) Let  $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$  be a strict partition such that  $\ell(\alpha) = 4$ . Let  $P = P_6(\alpha; f)$  the banner associated with  $\alpha$  and  $f$ . If  $\pi = (\sigma; \rho, \theta)$  is a  $P$ -partition satisfying the condition (1.10), then we have

$$W_P(\pi; q, t) = \widehat{\Phi}_1^f(\rho, \theta; q, t) b_{\sigma[0]}^{\text{el}}(q, t) \psi_{[\sigma[0], \dots, \sigma[n]]}^{\epsilon(\alpha)}(q, t) \quad (3.5)$$

and

$$z^\pi = (\widetilde{x}_2 w)^{\sigma_{11} + \sigma_{33}} \prod_{i=2}^f \widetilde{x}_i^{\rho_i + \theta_i - \rho_{i-1} - \theta_{i-1}} \prod_{i=1}^n \widetilde{z}_i^{\epsilon_i(\alpha) |\sigma[i-1] - \sigma[i]|_{\epsilon_i(\alpha)}}, \quad (3.6)$$

where  $w$ ,  $\widetilde{x}_i$  ( $1 \leq i \leq f$ ), and  $\widetilde{z}_i$  ( $1 \leq i \leq n$ ) are as in Proposition 1.11 (3).

**Proof.** (1) From (1.19) and (2.8) we have

$$f_\alpha^{\text{ND}}(\pi; q, t) = \begin{cases} \prod_{1 \leq i \leq j} \frac{f(\pi[1]_i - \pi[0]_{j+1}; j-i)}{f(\pi[1]_i - \pi[1]_j; j-i)} \prod_{i=2}^n \psi_{[\pi[i-1], \pi[i]]}^{\epsilon_i(\alpha)}(q, t) & \text{if } \epsilon_1(\alpha) = +, \\ \prod_{1 \leq i \leq j} \frac{f(\pi[1]_i - \pi[0]_{j+1}; j-i)}{f(\pi[1]_i - \pi[1]_j; j-i)} \prod_{i=2}^n \psi_{[\pi[i-1], \pi[i]]}^{\epsilon_i(\alpha)}(q, t) & \text{if } \epsilon_1(\alpha) = -. \end{cases}$$

Similarly, from (1.20) and (2.2) we have

$$f_\alpha^{\text{D}}(\pi; q, t) = \begin{cases} \prod_{1 \leq i \leq j} \frac{f(\pi[0]_i - \pi[1]_{j+1}; j-i)}{f(\pi[0]_i - \pi[0]_{j+1}; j-i)} b_{\pi[0]}^{\text{el}}(q, t) & \text{if } \epsilon_1(\alpha) = +, \\ \prod_{1 \leq i \leq j} \frac{f(\pi[0]_i - \pi[1]_{j+1}; j-i)}{f(\pi[0]_i - \pi[0]_{j+1}; j-i)} b_{\pi[0]}^{\text{el}}(q, t) & \text{if } \epsilon_1(\alpha) = -. \end{cases}$$

Hence we obtain (3.1) from (1.23) since

$$\psi_{[\pi[0], \pi[1]]}^{\epsilon_1(\alpha)}(q, t) = \begin{cases} \prod_{1 \leq i \leq j} \frac{f(\pi[0]_i - \pi[1]_{j+1}; j-i) f(\pi[1]_i - \pi[0]_{j+1}; j-i)}{f(\pi[1]_i - \pi[1]_j; j-i) f(\pi[0]_i - \pi[0]_{j+1}; j-i)} & \text{if } \epsilon_1(\alpha) = +, \\ \prod_{1 \leq i \leq j} \frac{f(\pi[1]_i - \pi[0]_{j+1}; j-i) f(\pi[0]_i - \pi[1]_{j+1}; j-i)}{f(\pi[1]_i - \pi[1]_j; j-i) f(\pi[0]_i - \pi[0]_{j+1}; j-i)} & \text{if } \epsilon_1(\alpha) = -. \end{cases}$$

Meanwhile, (3.2) can be easily obtained from

$$z^\pi = w^{\pi_{r-1, r-1} + \pi_{r-3, r-3} + \dots} \prod_{i=1}^n \widetilde{z}_i^{|\pi[i-1] - \pi[i]|}.$$

- (2) As in (1) we have

$$\begin{aligned} & f_\alpha^{\text{ND}}(\sigma; q, t) \\ &= \begin{cases} f(\sigma_{12} - \sigma_{11}; 0) \prod_{i=2}^n \psi_{[\sigma[i-1], \sigma[i]]}^{\epsilon_i(\alpha)}(q, t) & \text{if } \epsilon_1(\alpha) = +, \\ \frac{f(\sigma_{23} - \sigma_{22}; 0) f(\sigma_{23} - \sigma_{11}; 1) f(\sigma_{12} - \sigma_{11}; 0)}{f(\sigma_{23} - \sigma_{12}; 1)} \prod_{i=2}^n \psi_{[\sigma[i-1], \sigma[i]]}^{\epsilon_i(\alpha)}(q, t) & \text{if } \epsilon_1(\alpha) = -. \end{cases} \end{aligned}$$

From (1.19) and (2.7) we have

$$\begin{aligned} & f_\beta^{\text{ND}}(\tau; q, t) \\ &= \begin{cases} f(\tau_{12} - \tau_{11}; 0) \prod_{i=2}^n \phi_{[\tau[i-1], \tau[i]]}^{\epsilon_i(\beta)}(q, t) & \text{if } \epsilon_1(\beta) = +, \\ \frac{f(\tau_{23} - \tau_{22}; 0) f(\tau_{23} - \tau_{11}; 1) f(\tau_{12} - \tau_{11}; 0)}{f(\tau_{23} - \tau_{12}; 0)} \prod_{i=2}^n \phi_{[\tau[i-1], \tau[i]]}^{\epsilon_i(\beta)}(q, t) & \text{if } \epsilon_1(\beta) = -. \end{cases} \end{aligned}$$

Hence, if we use (2.7) or (2.8), then we obtain (3.3) from (1.24). On the other hand, (3.4) is easily obtained from

$$z^\pi = z_0^{\sigma_{11} + \sigma_{22}} \prod_{i=1}^f x_i^{\rho_i + \theta_i} \prod_{i=1}^m \tilde{z}_i^{\epsilon_i(\alpha) |\sigma[i-1] - \sigma[i]|_{\epsilon_i(\alpha)}} \prod_{i=1}^n \tilde{y}_i^{\epsilon_i(\beta) |\tau[i-1] - \tau[i]|_{\epsilon_i(\beta)}}$$

using  $z_0^{\sigma_{11} + \sigma_{22}} \prod_{i=0}^f x_i^{\rho_i + \theta_i} = (z_0 \tilde{x}_1)^{\rho_0 + \theta_0} \prod_{i=1}^f \tilde{x}_i^{\rho_i + \theta_i - \rho_{i-1} - \theta_{i-1}}$ , where we use the convention  $\sigma_{11} = \rho_0$  and  $\sigma_{22} = \theta_0$ .

(3) As in (1) we have

$$f_\alpha^D(\sigma; q, t) f_\alpha^{\text{ND}}(\sigma; q, t) = b_{\sigma[0]}^{\text{el}}(q, t) \psi_{[\sigma[0], \dots, \sigma[n]]}^{\epsilon(\alpha)}(q, t).$$

Hence we can obtain (3.5). Meanwhile, (3.6) can be obtained from

$$z^\pi = \prod_{i=2}^f x_i^{\rho_i + \theta_i} \cdot \left( \frac{z_0}{y_0} \right)^{\sigma_{11} + \sigma_{33}} \prod_{i=1}^n \tilde{z}_i^{|\pi[i-1]| - |\pi[i]|}$$

using  $\prod_{i=2}^f x_i^{\rho_i + \theta_i} = \tilde{x}_2^{\rho_1 + \theta_1} \prod_{i=2}^f \tilde{x}_i^{\rho_i + \theta_i - \rho_{i-1} - \theta_{i-1}}$ , where we use the convention  $\rho_1 = \sigma_{11}$  and  $\theta_1 = \pi_{33}$ .  $\square$

**Theorem 3.2.** (1) Let  $P = P_2(\alpha)$  be the shifted shape associated with a strict partition  $\alpha$  of length  $r$ . Let  $n$  be an integer such that  $n \geq \alpha_1$ , and let  $\alpha^c$  be the strict partition formed by the complement of  $\alpha$  in  $[n]$ . Then we have

$$\sum_{\pi \in \mathcal{A}(P)} W_P(\pi; q, t) z^\pi = \prod_{\alpha_k^c < \alpha_l} F\left(\tilde{z}_{\alpha_k^c}^{-1} \tilde{z}_{\alpha_l}\right) \sum_{\lambda} w^{\frac{|\lambda| - r(\lambda')}{2}} b_{\lambda}^{\text{el}}(q, t) P_{\lambda}(\tilde{z}_{\alpha_1} \dots, \tilde{z}_{\alpha_r}; q, t), \quad (3.7)$$

where  $w$  and  $\tilde{z}_i$  ( $i = 1, \dots, n$ ) are as in Proposition 1.11 (1).

(2) Let  $\alpha = (\alpha_1, \alpha_2)$  and  $\beta = (\beta_1, \beta_2)$  be strict partitions such that  $\ell(\alpha) = \ell(\beta) = 2$ . Let  $f > 0$  be a positive integer, and set  $P = P_3(\alpha, \beta; f)$  to be the bird associated with  $\alpha, \beta$  and  $f$ . Let  $m$  (resp.  $n$ ) be a positive integer such that  $m \geq \alpha_1$  (resp.  $n \geq \beta_1$ ). If  $\pi = (\sigma, \tau; \rho, \theta)$  is a  $P$ -partition satisfying the condition (1.9), then we have

$$\begin{aligned} \sum_{\pi \in \mathcal{A}(P)} W_P(\pi; q, t) z^\pi &= \prod_{\alpha_i^c < \alpha_j} F\left(\tilde{z}_{\alpha_i^c}^{-1} \tilde{z}_{\alpha_j}\right) \prod_{\beta_i^c < \beta_j} F\left(\tilde{y}_{\beta_i^c}^{-1} \tilde{y}_{\beta_j}\right) \\ &\times \sum_{(\rho, \theta)} \tilde{\Phi}_0^f(\tilde{x}; \rho, \theta; q, t) P_{(\theta_0, \rho_0)}(\tilde{x}_0 \tilde{z}_{\alpha_1}, \tilde{x}_0 \tilde{z}_{\alpha_2}; q, t) Q_{(\theta_0, \rho_0)}(\tilde{y}_{\beta_1}, \tilde{z}_{\beta_2}; q, t). \end{aligned} \quad (3.8)$$

where the sum on the right-hand side is taken over all pairs  $(\rho, \theta)$  with  $\rho = (\rho_0, \dots, \rho_f)$  and  $\theta = (\theta_0, \dots, \theta_f)$  satisfying

$$0 \leq \rho_f \leq \dots \leq \rho_0 \leq \theta_0 \leq \dots \leq \theta_f. \quad (3.9)$$

Here  $\tilde{x}_i$  ( $0 \leq i \leq f$ ),  $y_i$  ( $1 \leq i \leq n$ ) and  $z_i$  ( $1 \leq i \leq m$ ) are as in Proposition 1.11 (2).

(3) Let  $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$  be a strict partition such that  $\ell(\alpha) = 4$ . Let  $P = P_6(\alpha; f)$  be the banner associated with  $\alpha$  and  $f$ . If  $\pi = (\sigma; \rho, \theta)$  is a  $P$ -partition satisfying the condition (1.10), then we have

$$\begin{aligned} \sum_{\pi \in \mathcal{A}(P)} W_P(\pi; q, t) &= \prod_{\lambda_k^c < \lambda_l} F\left(\tilde{z}_{\lambda_k^c}^{-1} \tilde{z}_{\lambda_l}\right) \sum_{(\lambda, \rho, \theta)} \tilde{\Phi}_1^f(\tilde{x}; \rho, \theta; q, t) \\ &\times (\tilde{x}_2 w)^{\lambda_2 + \lambda_4} b_{\lambda}^{\text{el}}(q, t) P_{\lambda}(\tilde{z}_{\alpha_1}, \tilde{z}_{\alpha_2}, \tilde{z}_{\alpha_3}, \tilde{z}_{\alpha_4}; q, t), \end{aligned} \quad (3.10)$$

where the sum on the right-hand side is taken over all triplets  $(\lambda, \rho, \theta)$  with  $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ ,  $\rho = (\rho_1, \dots, \rho_f)$  and  $\theta = (\theta_1, \dots, \theta_f)$  satisfying

$$\lambda_4 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1, \quad 0 \leq \rho_f \leq \dots \leq \rho_1 = \lambda_4, \quad \lambda_2 = \theta_1 \leq \dots \leq \theta_f. \quad (3.11)$$

Here  $w, \tilde{x}_i$  ( $1 \leq i \leq f$ ) and  $z_i$  ( $1 \leq i \leq n$ ) are as in Proposition 1.11 (3).

**Proof.** (1) Since

$$\begin{aligned} \psi_{\pi[i-1]/\pi[i]}(q, t) \tilde{z}_i^{|\pi[i-1]-\pi[i]|} &= P_{\pi[i-1]/\pi[i]}(\tilde{z}_i; q, t), \\ \phi_{\pi[i]/\pi[i-1]}(q, t) \tilde{z}_i^{-|\pi[i-1]-\pi[i]|} &= Q_{\pi[i]/\pi[i-1]}(\tilde{z}_i^{-1}; q, t) \end{aligned}$$

(see [7, VI.7, (7.14)(7.14')]), we can use (2.17) to take the sum of the product of (3.1) and (3.2), then we obtain

$$\begin{aligned} &\sum_{\pi} W_P(\pi; q, t) z^{\pi} \\ &= \prod_{\alpha_k^c < \alpha_l} F\left(\tilde{z}_{\alpha_k^c}^{-1} \tilde{z}_{\alpha_l}\right) \sum_{\pi[0]} b_{\pi[0]}^{\text{el}}(q, t) w^{(|\pi[0]|-r(\pi[0]'))/2} P_{\pi[0]}(\tilde{z}_{\alpha_1}, \dots, \tilde{z}_{\alpha_r}; q, t), \end{aligned}$$

where the sum on the right-hand side runs over all partitions  $\pi[0]$ .

(2) Again, using (2.17) to take the sum of the product of (3.3) and (3.4), we obtain

$$\begin{aligned} \sum_{\pi} W_P(\pi; q, t) z^{\pi} &= \prod_{\alpha_k^c < \alpha_l} F\left(\tilde{z}_{\alpha_k^c}^{-1} \tilde{z}_{\alpha_l}\right) \prod_{\beta_k^c < \beta_l} F\left(\tilde{y}_{\beta_k^c}^{-1} \tilde{y}_{\beta_l}\right) \tilde{x}_0^{\rho_0 + \theta_0} \\ &\times \sum_{(\rho, \theta)} \tilde{\Phi}_0^f(\rho, \theta) \tilde{x}_0^{\rho_0 + \theta_0} P_{\sigma[0]}(\tilde{z}_{\alpha_1}, \tilde{z}_{\alpha_2}; q, t) Q_{\tau[0]}(\tilde{y}_{\beta_1}, \tilde{y}_{\beta_2}; q, t), \end{aligned}$$

where the sum on the right-hand side runs over all pairs  $(\rho, \theta)$  satisfying (3.9) with  $\sigma[0] = \tau[0] = (\theta_0, \rho_0)$ . Finally we use  $\tilde{x}_0^{\rho_0 + \theta_0} P_{(\theta_0, \rho_0)}(\tilde{z}_{\alpha_1}, \tilde{z}_{\alpha_2}; q, t) = P_{(\theta_0, \rho_0)}(\tilde{x}_0 \tilde{z}_{\alpha_1}, \tilde{x}_0 \tilde{z}_{\alpha_2}; q, t)$ .

(3) Using (2.17) to take the sum of the product of (3.5) and (3.6), we obtain

$$\begin{aligned} \sum_{\pi} W_P(\pi; q, t) z^{\pi} &= \prod_{\alpha_k^c < \alpha_l} F\left(\tilde{z}_{\alpha_k^c}^{-1} \tilde{z}_{\alpha_l}\right) \sum_{(\rho, \theta)} \tilde{\Phi}_1^f(\sigma[0], \rho, \theta) \\ &\times (\tilde{x}_2 w)^{\pi[0]_2 + \pi[0]_4} b_{\pi[0]}^{\text{el}}(q, t) w^{(|\sigma[0]|-r(\sigma[0]'))/2} P_{\sigma[0]}(\tilde{z}_{\alpha_1}, \tilde{z}_{\alpha_2}, \tilde{z}_{\alpha_3}, \tilde{z}_{\alpha_4}; q, t), \end{aligned}$$

where the sum on the right-hand side runs over all triplets  $(\sigma[0], \rho, \theta)$  satisfying (3.11).  $\square$

If we apply Warner's formula (2.23) to (3.7) we can obtain the  $(q, t)$ -hook formula (1.26) for shifted shapes. This gives another proof of [8, Proposition 4.5 (b)]. Now we look at the right-hand side of the conjectured identities in the cases of Birds and Banners. From Proposition 1.11 we can derive the following theorem.

**Theorem 3.3.** (1) Let  $\alpha = (\alpha_1, \alpha_2)$  and  $\beta = (\beta_1, \beta_2)$  be strict partitions of length 2. Let  $f > 0$  be a positive integer, and set  $P = P_3(\alpha, \beta; f)$  the bird associated with  $f, \alpha$  and  $\beta$ . Let  $m, n$  be integers such that  $m \geq \ell(\alpha)$  and  $n \geq \ell(\beta)$ , and let  $\alpha^c$  (resp.  $\beta^c$ ) be the strict partition formed by the complement of  $\alpha$  (resp.  $\beta$ ) in  $[m]$  (resp.  $[n]$ ). Then we have

$$\begin{aligned} F(z[H_P]; q, t) &= \prod_{\alpha_i^c < \alpha_j} F\left(\tilde{z}_{\alpha_i^c}^{-1} \tilde{z}_{\alpha_j}\right) \prod_{\beta_i^c < \beta_j} F\left(\tilde{y}_{\beta_i^c}^{-1} \tilde{y}_{\beta_j}\right) \\ &\times \sum_{\substack{\lambda \\ \ell(\lambda) \leq 2}} \sum_{l=0}^{\lambda_2} \sum_{k_1, \dots, k_f \geq 0} \sum_{\substack{l_1, \dots, l_f \geq 0 \\ l_1 + \dots + l_f = l}} \prod_{i=1}^f f(k_i, 0) f(l_i, 0) \tilde{x}_i^{k_i - l_i} \\ &\times \frac{b_{\lambda - l \cdot 1^2}(q, t)}{b_{\lambda}(q, t)} P_{\lambda}(\tilde{x}_1 \tilde{z}_{\alpha_1}, \tilde{x}_1 \tilde{z}_{\alpha_2}; q, t) Q_{\lambda}(\tilde{y}_{\beta_1}, \tilde{y}_{\beta_2}; q, t) \end{aligned} \quad (3.12)$$

where  $\tilde{x}_i$  ( $1 \leq i \leq f$ ),  $\tilde{y}_i$  ( $1 \leq i \leq n$ ) and  $\tilde{z}_i$  ( $1 \leq i \leq m$ ) are as in Proposition 1.11 (2).

- (2) Let  $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$  be a strict partition of length 4. Let  $P = P_6(f; \alpha)$  the Banner associated with  $\alpha$  and  $\beta$ . Let  $n$  be an integer such that  $n \geq 4 = \ell(\alpha)$ , and let  $\alpha^c$  be the strict partition formed by the complement of  $\alpha$  in  $[n]$ . We write  $y_0 = z_{0'}$  and  $x_i = z_{-i}$  for  $i = 1, \dots, f$ . Then we have

$$\begin{aligned} F(z[H_p]; q, t) &= \prod_{\alpha_i^c < \alpha_j} F\left(\tilde{z}_{\alpha_i^c}^{-1} \tilde{z}_{\alpha_j}\right) \\ &\times \sum_{\substack{\lambda \\ \ell(\lambda) \leq 4}} \sum_{l=0}^{\lambda_4} \sum_{k_2, \dots, k_f \geq 0} \sum_{\substack{l_2, \dots, l_f \geq 0 \\ l_2 + \dots + l_f = l}} \prod_{i=2}^f f(k_i; 0) f(l_i; 0) \tilde{x}_i^{k_i - l_i} \\ &\times (\tilde{x}_2 w)^{\lambda_2 + \lambda_4} b_{\lambda - l, 1^4}^{\text{el}}(q, t) P_\lambda(\tilde{z}_{\alpha_1}, \tilde{z}_{\alpha_2}, \tilde{z}_{\alpha_3}, \tilde{z}_{\alpha_4}; q, t) \end{aligned} \quad (3.13)$$

where  $w, \tilde{x}_i$  ( $2 \leq i \leq f$ ) and  $\tilde{z}_i$  ( $1 \leq i \leq n$ ) are as in Proposition 1.11 (3).

**Proof.** (1) From (2.6) we have

$$\prod_{i,j=1}^2 F(\tilde{x}_1 \tilde{y}_{\beta_j} \tilde{z}_{\alpha_i}) = \sum_{\mu} P_{\mu}(\tilde{x}_1 \tilde{z}_{\alpha_1}, \tilde{x}_1 \tilde{z}_{\alpha_2}) Q_{\mu}(\tilde{y}_{\beta_1}, \tilde{y}_{\beta_2}).$$

By the binomial theorem we have

$$\begin{aligned} \prod_{i=1}^f F(\tilde{x}_i) &= \sum_{k_1, \dots, k_f \geq 0} \prod_{i=1}^f f(k_i; 0) \tilde{x}_i^{k_i}, \\ \prod_{i=1}^f F\left(\frac{\tilde{x}_1^2}{\tilde{x}_i} \prod_{k,l=1}^2 \tilde{y}_l \tilde{z}_k\right) &= \sum_{l_1, \dots, l_f \geq 0} \prod_{i=1}^f f(l_i; 0) \tilde{x}_i^{-l_i} \left(\tilde{x}_1^2 \prod_{k,l=1}^2 \tilde{y}_l \tilde{z}_k\right)^{l_1 + \dots + l_f}. \end{aligned}$$

By [7, VI.4, (4.17)] and (2.5) we obtain

$$\begin{aligned} (\tilde{x}_1^2 \tilde{z}_1 \tilde{z}_2)^l P_{\mu}(\tilde{x}_1 \tilde{z}_{\alpha_1}, \tilde{x}_1 \tilde{z}_{\alpha_2}) &= P_{\mu+l, 1^2}(\tilde{x}_1 \tilde{z}_{\alpha_1}, \tilde{x}_1 \tilde{z}_{\alpha_2}), \\ (\tilde{y}_1 \tilde{y}_2)^l Q_{\mu}(\tilde{y}_{\beta_1}, \tilde{y}_{\beta_2}) &= \frac{b_{\mu}(q, t)}{b_{\mu+l, 1^2}(q, t)} Q_{\mu+l, 1^2}(\tilde{y}_{\beta_1}, \tilde{y}_{\beta_2}). \end{aligned}$$

From (1.27) we obtain

$$\begin{aligned} F(z[H_p]; q, t) &= \prod_{\alpha_i^c < \alpha_j} F\left(\tilde{z}_{\alpha_i^c}^{-1} \tilde{z}_{\alpha_j}\right) \prod_{\beta_i^c < \beta_j} F\left(\tilde{y}_{\beta_i^c}^{-1} \tilde{y}_{\beta_j}\right) \\ &\times \sum_{l \geq 0} \sum_{\substack{\mu \\ \ell(\mu) \leq 2}} \sum_{k_1, \dots, k_f \geq 0} \sum_{\substack{l_1, \dots, l_f \geq 0 \\ l_1 + \dots + l_f = l}} \prod_{i=1}^f f(k_i; 0) f(l_i; 0) \tilde{x}_i^{k_i - l_i} \\ &\times \frac{b_{\mu}(q, t)}{b_{\mu+l, 1^2}(q, t)} P_{\mu+l, 1^2}(\tilde{x}_1 \tilde{z}_{\alpha_1}, \tilde{x}_1 \tilde{z}_{\alpha_2}; q, t) Q_{\mu+l, 1^2}(\tilde{y}_{\beta_1}, \tilde{y}_{\beta_2}; q, t). \end{aligned}$$

This immediately implies (3.12).

- (2) From Warner's formula (2.23), we have

$$\begin{aligned} \prod_{i=1}^4 F(\tilde{z}_{\alpha_i}; q, t) &\prod_{1 \leq i < j \leq 4} F(w \tilde{x}_2 \tilde{z}_{\alpha_i} \tilde{z}_{\alpha_j}; q, t) \\ &= \sum_{\mu} w^{\mu_2 + \mu_4} b_{\mu}^{\text{el}}(q, t) P_{\mu}(\tilde{z}_{\alpha_1}, \tilde{z}_{\alpha_2}, \tilde{z}_{\alpha_3}, \tilde{z}_{\alpha_4}; q, t). \end{aligned}$$

By the binomial theorem we have

$$\begin{aligned} \prod_{i=2}^f F(\tilde{x}_i) &= \sum_{k_2, \dots, k_f \geq 0} \prod_{i=2}^f f(k_i; 0) \tilde{x}_i^{k_i}, \\ \prod_{i=2}^f F\left(\frac{\tilde{x}_2^2}{\tilde{x}_i} w^2 \prod_{i=1}^4 \tilde{z}_{\alpha_i}\right) &= \sum_{l_2, \dots, l_f \geq 0} \prod_{i=2}^f f(l_i; 0) \tilde{x}_i^{-l_i} \left(\tilde{x}_2^2 w^2 \prod_{i=1}^4 \tilde{z}_{\alpha_i}\right)^{l_1 + \dots + l_f}. \end{aligned}$$

From (1.27) we obtain

$$\begin{aligned} F(z[H_p]; q, t) &= \prod_{\alpha_i^c < \alpha_j} F(\tilde{z}_{\alpha_i^c}^{-1} \tilde{z}_{\alpha_j}) \\ &\times \sum_{l \geq 0} \sum_{\ell(\mu) \leq 4} \sum_{k_2, \dots, k_f \geq 0} \sum_{\substack{l_2, \dots, l_f \geq 0 \\ l_2 + \dots + l_f = l}} \prod_{i=2}^f f(k_i; 0) f(l_i; 0) \tilde{x}_i^{k_i - l_i} \\ &\times (\tilde{x}_2 w)^{\mu_2 + \mu_4 + 2l} b_{\mu}^{\text{el}}(q, t) P_{\mu + l, 1^4}(\tilde{z}_{\alpha_1}, \tilde{z}_{\alpha_2}, \tilde{z}_{\alpha_3}, \tilde{z}_{\alpha_4}; q, t). \end{aligned}$$

This immediately implies (3.12).  $\square$

## 4 Proof by Gasper's formula

Now we are in position to prove Okada's conjecture for birds and banners, i.e., Theorem 1.9. At the last step of our proof Gasper's identity (1.2) plays an important role.

We use the fact that Macdonald's polynomials are basis for  $\Lambda_{\mathbb{F}}$ . (cf. [6]). To prove the birds case, we fix integers  $\rho_0$  and  $\theta_0$  such that  $\theta_0 \geq \rho_0 \geq 0$ , and nonnegative integers  $r_1, \dots, r_f$ . If we compare the coefficient of  $\prod_{i=1}^f \tilde{x}_i^{r_i} \cdot P_{\lambda}(\tilde{x}_1 \tilde{z}_{\alpha_1}, \tilde{x}_1 \tilde{z}_{\alpha_2}; q, t) Q_{\lambda}(\tilde{y}_{\beta_1}, \tilde{y}_{\beta_2}; q, t)$  in (3.8) and (3.12), the following identity must hold:

$$\sum_{\substack{(\rho_1, \dots, \rho_f) \\ 0 \leq \rho_f \leq \dots \leq \rho_1 \leq \rho_0}} \hat{\Phi}_0^f(\rho, \theta; q, t) = \sum_{l=0}^{\rho_0} \sum_{\substack{l_1, \dots, l_f \geq 0 \\ l_1 + \dots + l_f = l}} \frac{b_{(\theta_0 - l, \rho_0 - l)}(q, t)}{b_{(\theta_0, \rho_0)}(q, t)} \prod_{i=1}^f f(l_i; 0) f(l_i + r_i; 0),$$

where  $(\theta_1, \dots, \theta_f)$  is determined from  $\theta_0$  and  $(\rho_1, \dots, \rho_f)$  by using the equations  $\theta_i = \rho_{i-1} + \theta_{i-1} + r_i - \rho_i$  for  $i = 1, \dots, f$ . Since (2.1) implies

$$b_{(\theta_0, \rho_0)} = f(\theta_0 - \rho_0; 0) \frac{f(\theta_0; 1)}{f(\theta_0 - \rho_0; 1)} f(\rho_0; 0),$$

we obtain

$$\frac{b_{(\theta_0 - l, \rho_0 - l)}(q, t)}{b_{(\theta_0, \rho_0)}(q, t)} = \frac{f(\rho_0 - l; 0) f(\theta_0 - l; 1)}{f(\rho_0; 0) f(\theta_0; 1)}.$$

Hence it is enough to prove

$$\sum_{\substack{(\rho_1, \dots, \rho_f) \\ 0 \leq \rho_f \leq \dots \leq \rho_1 \leq \rho_0}} \hat{\Phi}_0^f(\rho, \theta; q, t) = \sum_{l=0}^{\rho_0} \sum_{\substack{l_1, \dots, l_f \geq 0 \\ l_1 + \dots + l_f = l}} \frac{f(\rho_0 - l; 0) f(\theta_0 - l; 1)}{f(\rho_0; 0) f(\theta_0; 1)} \prod_{i=1}^f f(l_i; 0) f(l_i + r_i; 0). \quad (4.1)$$

In the case of banners we fix a partition  $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$  of length 4 and nonnegative integers  $r_2, \dots, r_f$ . If we compare the coefficient of  $\prod_{i=2}^f \tilde{x}_i^{r_i} \cdot P_{\lambda}(\tilde{x}_1 \tilde{z}_{\alpha_1}, \tilde{x}_1 \tilde{z}_{\alpha_2}; q, t)$  in (3.10) and (3.13), the following identity must hold:

$$\sum_{\substack{(\rho_2, \dots, \rho_f) \\ 0 \leq \rho_f \leq \dots \leq \rho_2 \leq \rho_1}} \hat{\Phi}_1^f(\rho, \theta; q, t) = \sum_{l=0}^{\lambda_4} \sum_{\substack{l_2, \dots, l_f \geq 0 \\ l_2 + \dots + l_f = l}} \frac{b_{\lambda - l, 1^4}^{\text{el}}(q, t)}{b_{\lambda}^{\text{el}}(q, t)} \prod_{i=2}^f f(l_i; 0) f(l_i + r_i; 0),$$

where  $(\theta_2, \dots, \theta_f)$  is determined from  $\theta_1$  and  $(\rho_2, \dots, \rho_f)$  by using the equations  $\theta_i = \rho_{i-1} + \theta_{i-1} + r_i - \rho_i$  for  $i = 2, \dots, f$ . Here we use the convention that  $\rho_1 = \lambda_4$  and  $\theta_1 = \lambda_2$ . Again, because of (2.2) we obtain

$$\frac{b_{\lambda-l, 1^4}^{\text{el}}(q, t)}{b_{\lambda}^{\text{el}}(q, t)} = \frac{f(\lambda_4 - l; 0)f(\lambda_2 - l; 2)}{f(\lambda_4; 0)f(\lambda_2; 2)}.$$

Hence it is enough to prove

$$\sum_{\substack{(\rho_2, \dots, \rho_f) \\ 0 \leq \rho_f \leq \dots \leq \rho_2 \leq \rho_1}} \widehat{\Phi}_1^f(\rho, \theta; q, t) = \sum_{l=0}^{\lambda_4} \sum_{\substack{l_2, \dots, l_f \geq 0 \\ l_2 + \dots + l_f = l}} \frac{f(\lambda_4 - l; 0)f(\lambda_2 - l; 2)}{f(\lambda_4; 0)f(\lambda_2; 2)} \prod_{i=2}^f f(l_i; 0)f(l_i + r_i; 0). \quad (4.2)$$

In fact a more general formula holds. If we prove the following theorem, then the proof of (4.1) and (4.2) are both done.

**Theorem 4.1.** Let  $m$  and  $n$  be nonnegative integers. Let  $k_0, \rho_0, \theta_0$  be integers such that  $0 \leq k_0 \leq \rho_0 \leq \theta_0$ , and let  $\gamma_1, \dots, \gamma_n$  be nonnegative integers. Then we have

$$\begin{aligned} & \sum_{\substack{(\rho_1, \dots, \rho_n) \\ k_0 \leq \rho_n \leq \dots \leq \rho_1 \leq \rho_0}} f(\rho_n - k_0; 0)f(\theta_n - k_0; m + n) \\ & \times \prod_{i=1}^n \frac{f(\rho_{i-1} - \rho_i; 0)f(\theta_{i-1} - \rho_i; i + m - 1)f(\theta_i - \rho_{i-1}; i + m - 1)f(\theta_i - \theta_{i-1}; 0)}{f(\theta_i - \rho_i; i + m - 1)f(\theta_i - \rho_i; i + m)} \\ & = \sum_{\substack{k_1, \dots, k_n \geq 0 \\ k_1 + \dots + k_n \leq \rho_0 - \rho_{m+1}}} f(\rho_0 - \sum_{i=0}^n k_i; 0)f(\theta_0 - \sum_{i=0}^n k_i; m) \prod_{i=1}^n f(k_i; 0)f(k_i + \gamma_i; 0), \end{aligned} \quad (4.3)$$

where the sum on the left-hand side runs over all  $n$ -tuples  $(\rho_1, \dots, \rho_n)$  of nonnegative integers such that  $k_0 \leq \rho_n \leq \dots \leq \rho_1 \leq \rho_0$ , the sum on the right-hand side runs over all  $n$ -tuples  $(k_1, \dots, k_n)$  of nonnegative integers which satisfy  $k_1 + \dots + k_n \leq \rho_0 - \rho_{m+1}$ , and  $\theta_i$  is determined from  $\rho_i, \rho_{i-1}$  and  $\theta_{i-1}$  by  $\theta_i = \gamma_i + \theta_{i-1} + \rho_{i-1} - \rho_i$  for  $i = 1, \dots, n$ .

Before we prove this theorem, we need the following lemma which is a special case (i.e.,  $n = 1$ ) of this theorem.

**Lemma 4.2.** Let  $m$  be a nonnegative integer. Let  $k_0, \rho_0$  and  $\theta_0$  be integers such that  $0 \leq k_0 \leq \rho_0 \leq \theta_0$ , and let  $\gamma$  be a nonnegative integer. Then we have

$$\begin{aligned} & \sum_{\rho=k_0}^{\rho_0} f(\rho - k_0; 0)f(\theta - k_0; m + 1) \frac{f(\rho_0 - \rho; 0)f(\theta_0 - \rho; m)f(\theta - \rho_0; m)f(\theta - \theta_0; 0)}{f(\theta - \rho; m)f(\theta - \rho; m + 1)} \\ & = \sum_{k=0}^{\rho_0 - k_0} f(\rho_0 - k_0 - k; 0)f(\theta_0 - k_0 - k; m)f(k; 0)f(k + \gamma; 0), \end{aligned} \quad (4.4)$$

where  $\theta = \gamma + \rho_0 + \theta_0 - \rho$ .

**Proof.** Set  $S_1$  to be the left-hand side of (4.4). If one puts  $k = \rho_0 - \rho$ , then  $\rho = \rho_0 - k$  and  $\theta = k + \gamma + \theta_0$ . Hence one obtains

$$\begin{aligned} S_1 &= \sum_{k=0}^{\rho_0 - k_0} f(\rho_0 - k_0 - k; 0)f(k + \gamma + \theta_0 - k_0; m + 1) \\ & \times \frac{f(k; 0)f(k + \gamma + \theta_0 - \rho_0; m)f(k + \theta_0 - \rho_0; m)f(k + \gamma; 0)}{f(2k + \gamma + \theta_0 - \rho_0; m)f(2k + \gamma + \theta_0 - \rho_0; m + 1)}. \end{aligned}$$

If we use

$$(\alpha; q)_{2k} = (\alpha^{\frac{1}{2}}; q)_k (-\alpha^{\frac{1}{2}}; q)_k (\alpha^{\frac{1}{2}} q^{\frac{1}{2}}; q)_k (-\alpha^{\frac{1}{2}} q^{\frac{1}{2}}; q)_k,$$



then the factors in the denominator are written as  $f(2k + \gamma + \theta_0 - \rho_0; m) = f(\gamma + \theta_0 - \rho_0; m) \times \frac{\left(t^{\frac{m+1}{2}} q^{\frac{\gamma+\theta_0-\rho_0}{2}}, -t^{\frac{m+1}{2}} q^{\frac{\gamma+\theta_0-\rho_0}{2}}, t^{\frac{m+1}{2}} q^{\frac{\gamma+\theta_0-\rho_0+1}{2}}, -t^{\frac{m+1}{2}} q^{\frac{\gamma+\theta_0-\rho_0+1}{2}}; q\right)_k}{\left(t^{\frac{m}{2}} q^{\frac{\gamma+\theta_0-\rho_0+1}{2}}, -t^{\frac{m}{2}} q^{\frac{\gamma+\theta_0-\rho_0+1}{2}}, t^{\frac{m}{2}} q^{\frac{\gamma+\theta_0-\rho_0+2}{2}}, -t^{\frac{m}{2}} q^{\frac{\gamma+\theta_0-\rho_0+2}{2}}; q\right)_k}$  and  $f(2k + \gamma + \theta_0 - \rho_0; m+1) = \frac{f(\gamma + \theta_0 - \rho_0; m+1) \times \left(t^{\frac{m+2}{2}} q^{\frac{\gamma+\theta_0-\rho_0}{2}}, -t^{\frac{m+2}{2}} q^{\frac{\gamma+\theta_0-\rho_0}{2}}, t^{\frac{m+2}{2}} q^{\frac{\gamma+\theta_0-\rho_0+1}{2}}, -t^{\frac{m+2}{2}} q^{\frac{\gamma+\theta_0-\rho_0+1}{2}}; q\right)_k}{\left(t^{\frac{m+1}{2}} q^{\frac{\gamma+\theta_0-\rho_0+1}{2}}, -t^{\frac{m+1}{2}} q^{\frac{\gamma+\theta_0-\rho_0+1}{2}}, t^{\frac{m+1}{2}} q^{\frac{\gamma+\theta_0-\rho_0+2}{2}}, -t^{\frac{m+1}{2}} q^{\frac{\gamma+\theta_0-\rho_0+2}{2}}; q\right)_k}$ . Meanwhile, the factors in the numerator are  $f(\rho_0 - k_0 - k; 0) = f(\rho_0 - k_0; 0) \frac{(q^{-\rho_0+k_0}; q)_k}{(t^{-1}q^{-\rho_0+k_0+1}; q)_k} \left(\frac{q}{t}\right)^k$ ,  $f(k + \gamma + \theta_0 - k_0; m+1) = f(\gamma + \theta_0 - k_0; m+1) \frac{(t^{m+2}q^{\gamma+\theta_0-k_0}; q)_k}{(t^{m+1}q^{\gamma+\theta_0-k_0+1}; q)_k}$ ,  $f(k + \gamma + \theta_0 - \rho_0; m) = f(\gamma + \theta_0 - \rho_0; m) \frac{(t^{m+1}q^{\gamma+\theta_0-\rho_0}; q)_k}{(t^mq^{\gamma+\theta_0-\rho_0+1}; q)_k}$ ,  $f(k + \theta_0 - \rho_0; m) = f(\theta_0 - \rho_0; m) \frac{(t^{m+1}q^{\theta_0-\rho_0}; q)_k}{(t^mq^{\theta_0-\rho_0+1}; q)_k}$ ,  $f(k + \gamma; 0) = f(k + \gamma; 0) \frac{(tq^\gamma; q)_k}{(q^\gamma; q)_k}$ . Hence, substituting these factors, we obtain

$$S_1 = C \cdot {}_{12}W_{11} \left( bc/d; (bcq/ad)^{\frac{1}{2}}, -(bcq/ad)^{\frac{1}{2}}, q(bc/d)^{\frac{1}{2}}, -q(bc/d)^{\frac{1}{2}}, \right. \\ \left. ab/d, ac/d, a, b, c; q, q/a \right),$$

where  $a = t$ ,  $b = tq^\gamma$ ,  $c = q^{-\rho_0+k_0}$ ,  $d = t^{-m}q^{-\theta_0+k_0}$  and

$$C = \frac{f(\rho_0 - k_0; 0)f(\gamma + \theta_0 - k_0; m+1)f(\theta_0 - \rho_0; m)f(\gamma; 0)}{f(\gamma + \theta_0 - \rho_0; m+1)}.$$

On the other hand, Set  $S_2$  to be the right-hand side of (4.4). If we use  $f(\rho_0 - k_0 - k; 0) = f(\rho_0 - k_0; 0) \frac{(q^{-\rho_0+k_0}; q)_k}{(t^{-1}q^{-\rho_0+k_0+1}; q)_k} \left(\frac{q}{t}\right)^k$ ,  $f(\theta_0 - k_0 - k; m) = f(\theta_0 - k_0; m) \frac{(t^{-m}q^{-\theta_0+k_0}; q)_k}{(t^{-m-1}q^{-\theta_0+k_0+1}; q)_k} \left(\frac{q}{t}\right)^k$  and  $f(k + \gamma; 0) = f(k + \gamma; 0) \frac{(tq^\gamma; q)_k}{(q^\gamma; q)_k}$ , then we obtain

$$S_2 = f(\rho_0 - k_0; 0)f(\theta_0 - k_0; m)f(\gamma; 0) {}_4\phi_3 \left[ \begin{matrix} q^{-\rho_0+k_0}, t^{-m}q^{-\theta_0+k_0}, t, tq^\gamma \\ t^{-1}q^{-\rho_0+k_0+1}, t^{-m-1}q^{-\theta_0+k_0+1}, q^{\gamma+1}; q, \frac{q^2}{t^2} \end{matrix} \right].$$

Hence Gasper's formula (1.2) proves that  $S_1 = S_2$ . The details are left to the reader. This completes our proof.  $\square$

**Proof of Theorem 4.1.** We proceed by induction on  $n$ . If  $n = 1$ , then (4.3) is nothing but (4.4). Let  $n \geq 2$  and assume (4.3) is true for  $n - 1$ . If we set  $S$  to be the left-hand side of (4.3), then we have

$$S = \sum_{\rho_1=k_0}^{\rho_0} \frac{f(\rho_0 - \rho_1; 0)f(\theta_0 - \rho_1; m)f(\theta_1 - \rho_0; m)f(\theta_1 - \theta_0; 0)}{f(\theta_1 - \rho_1; m)f(\theta_1 - \rho_1; m+1)} \\ \times \sum_{\substack{(\rho_2, \dots, \rho_n) \\ k_0 \leq \rho_n \leq \dots \leq \rho_2 \leq \rho_1}} f(\rho_n - k_0; 0)f(\theta_n - k_0; m+n) \\ \times \prod_{i=2}^n \frac{f(\rho_{i-1} - \rho_i; 0)f(\theta_{i-1} - \rho_i; i+m-1)f(\theta_i - \rho_{i-1}; i+m-1)f(\theta_i - \theta_{i-1}; 0)}{f(\theta_i - \rho_i; i+m-1)f(\theta_i - \rho_i; i+m)}.$$

We can use our induction hypothesis to obtain

$$S = \sum_{\substack{k_2, \dots, k_n \geq 0 \\ k_2 + \dots + k_n \leq \rho_0 - k_0}} \prod_{i=2}^n f(k_i; 0)f(k_i + \gamma_i; 0) \\ \times \sum_{\rho_1=k_0+\sum_{i=2}^n k_i}^{\rho_0} \frac{f(\rho_1 - k_0 - \sum_{i=2}^n k_i; 0)f(\theta_1 - k_0 - \sum_{i=2}^n k_i; m+1)}{f(\rho_0 - \rho_1; 0)f(\theta_0 - \rho_1; m)f(\theta_1 - \rho_0; m)f(\theta_1 - \theta_0; 0)}.$$

If we use (4.4) again, then we obtain

$$S = \sum_{\substack{k_2, \dots, k_n \geq 0 \\ k_2 + \dots + k_n \leq \rho_0 - k_0}} \prod_{i=2}^n f(k_i; 0) f(k_i + \gamma_i; 0) \\ \times \sum_{0 \leq k_1 \leq \rho_0 - k_0 - \sum_{i=2}^n k_i} f(\rho_0 - \sum_{i=0}^n k_i, 0) f(\theta_0 - \sum_{i=0}^n k_i, m) f(k_1, 0) f(k_1 + \gamma_1, 0),$$

which equals the right-hand side of (4.3). This completes our proof.  $\square$

**Concluding Remarks** In the proof of the  $(q, t)$  hook formula for birds and banners, Gasper's identity (1.2) for  ${}_{12}W_{11}$  plays an important role. The author tried the other classes of irreducible  $d$ -complete posets, but it seems that another identity will be needed for the rest.

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